We analyse the decay $\tau \to \pi \pi \pi \nu$ based on the recently developed techniques to generate axial-vector resonances dynamically. Under the assumption that the $a_1$ is generated entirely through coupled-channel dynamics, one gets a reasonable description of the spectral function. Including, in addition, an explicit elementary $a_1$ does not improve the results.

© 2008 Elsevier B.V. All rights reserved.
Before going into the details of the calculations, we want to put our work in the broader context of the existing literature. Unitary extensions of chiral perturbation theory have shed new light on the structure of several resonances. The low-lying scalars \((\sigma, \phi_0(980), a_0(980), \kappa(900))\), for example, appear as bound states in such calculations \([14,15]\). Similar works have been done in the meson–baryon sector \(\text{see e.g. } [16–18] \text{ and references therein}\), which suggest a number of \(J^P = \frac{1}{2}^-\) baryon resonances to be generated dynamically, in particular the \(\Lambda(1405)\) and \(N^*(1535)\). It is interesting in that context that recent studies \([19]\) find another interpretation of the very same results. In these studies it is proposed how to distinguish dynamically generated resonances from genuine quark states. According to \([19]\) the \(N^*(1535)\) also requires a genuine three-quark component in addition to an important contribution from dynamical generation. Studying the interaction of the pseudoscalar mesons with the decuplet of baryons \([20,21]\) also led to the generation of many known \(J^P = \frac{3}{2}^-\) resonances, \(\text{e.g. the } \Lambda(1520)\). Recent works applied the approach to the interactions of the octet of pseudoscalar mesons with the nonet of vector mesons \([5,6]\), as already mentioned. The only free parameter in the calculation enters through the regularisation of the loop integral in the Bethe–Salpeter equation. Poles have been found, which have been attributed to the axial-vector mesons, but no direct comparison to decay and scattering data has been performed. A comparison of the pole position and width is necessarily indirect and depends on the model, which is used to extract these quantities from the actual observable quantities. In addition, the height of the scattering amplitude, or in other words, the strength of the interaction, is not tested in this way. In the following we use this framework to describe the final state interaction of Goldstone boson and vector meson produced in \(\tau\) decays.

The \(a_1\) is especially interesting as it is considered to be the chiral partner of the \(\rho\). One expects a chiral partner for every particle from chiral symmetry. Due to the spontaneous symmetry breaking, one does not find degenerate one-particle states with the appropriate quantum numbers. Nevertheless, the chiral partners have to exist, not necessarily as one-particle states, but at least as multiparticle states. Unmasking the \(a_1\) as a bound state of a vector meson with a Goldstone boson would therefore approve its role as the chiral partner. In the meson–meson and meson–baryon scattering examples mentioned before, one can also see that some of the dynamically generated resonances would qualify as the chiral partners of the scattered particles, although the question of the chiral partners for these particles is not as clear as for the \(a_1\) and the \(\rho\). In addition, the \(a_1\) is accessible by a process, which is free of hadronic initial state interactions in contrast to \(\sigma\) or \(N^*(1535)\). Hence, more solid conclusions can be drawn since one has on the one hand good quality data and, on the other hand, a framework in which both scenarios \((a_1\) as a hadronic bound-state vs. \(a_1\) as an elementary state) can be explored. The energies involved in the \(\tau\) decay are well beyond \(1\) GeV, and the decay is dominated by resonance structures. Thus, we cannot expect pure CHPT to work for the whole energy region, covered by the \(\tau\) decay \(\text{see e.g. } [22]\). Including the vector mesons at tree level, one can also not produce an axial resonance. In case we are not including the \(a_1\) explicitly, the picture we promote is that the decay is dominated by the decay into Goldstone boson and vector meson. The structure, which is usually attributed to the \(a_1\), is generated by the rescattering process of the vector meson and the Goldstone boson. Since there are two channels with the quantum numbers of the \(a_1\), namely \(\rho \sigma\) and \(K^*K\), one has to solve a coupled channel problem.

The processes, we include in the first calculation, correspond to the upper six diagrams in Fig. 1. The first two diagrams are the lowest order contributions from CHPT, the next two correspond to the lowest order diagrams including the vector mesons and the fifth and sixth diagram describe the rescattering process, which is driven by the WT term \(\text{see Fig. 2}\). The couplings describing the decay of the \(W\)-boson into vector meson and Goldstone boson are related by chiral symmetry breaking to the decay of the \(\rho\) into dileptons and two pions, respectively. We refer to \([23,24]\) for further details and in particular for the definition of the coupling constants. In the present work we use \(f_V = 0.154 \text{ GeV}/M_\rho\) for the coupling of the \(\rho\) to a photon and \(g_V = 0.069 \text{ GeV}/M_\rho\) for the coupling of the \(\rho\) to two pions. These numbers are based on the experimental values for the mentioned decays. Thus, all coupling constants are fixed by chiral symmetry breaking and the properties of the \(\rho\). We note that in \([23]\) the authors also give a theoretical estimate for \(f_V\) and \(g_V\), which slightly deviates from the measured values. The dependence of the results on this choice will be discussed elsewhere \([25]\).

Using vector fields as interpolating fields for the vector mesons leads to an unreasonable high-energy behaviour of the spectral function for the \(\tau\) decay. Thus, we included higher order corrections to cure the high-energy behaviour \([23,26]\), which means that we additionally include \(O(q^4)\) expressions describing the direct decay into three pions. Here and in the following \(q\) is the momentum of the Goldstone bosons in a chiral counting.

For this work, we took the scattering amplitude from \([5]\). We also investigated the dependence of the results on this choice, by taking, for example, the scattering amplitude from \([6]\), which will be discussed in \([25]\). Of course, the loops which appear in Figs. 1 and 2 need renormalisation. In \([5]\) the authors use crossing...
symmetry arguments in order to fix the subtraction point $\mu_1$ of the loop diagrams (see Fig. 2) at $\mu_1 = M^2_\rho$. Based on different arguments the same value has been proposed in [19] to avoid contamination of the scattering amplitude from a genuine quark-antiquark state. Indeed, according to the arguments of [19] a higher value of $\mu_1$ indicates the presence of a preformed, i.e. not dynamically generated state.

Looking at the third row of diagrams in Fig. 1, one sees that one encounters an additional loop diagram, which is the first loop, containing the decay vertex of the $W$-boson. There is no reason to choose the subtraction point $\mu_2$ for that loop the same as for the loops in the scattering amplitude. Changing $\mu_2$ acts as a higher order correction to the vertex of the reaction $W$-boson to hadrons and is independent of the scattering amplitude. In Fig. 3 we show the results for different values of $\mu_2$ keeping $\mu_1 = M^2_\rho$. One sees that a peak appears for all values of the only free parameter $\mu_2$. Adjusting $\mu_2$ one influences the height and the width of the peak at the same time, and the choice $\mu_2 = 8.5 M^2_\rho$ reproduces the qualitative features of the data very well and also gives an approximately quantitative description. We note that the subtraction point at $8.5 M^2_\rho$ roughly corresponds to a cutoff of 1 GeV in a cutoff scheme.

In the calculation, we show, we folded the two-particle propagator in the loops with a spectral function (taken from [27]) for the vector particles. In other words, the mass squared of the zero width $\rho$ is treated as a variable in the loops. Each loop is multiplied by the spectral function and integrated over up to a cutoff in order to account for the normalisation. Including the spectral distribution of the vector particles leads to a small broadening of the peak and a small shift to the right. In addition, it smoothen the kink resulting from the threshold effect of the $K^+K^-$ channel [25].

Neglecting the coupled channel structure and considering only the $\rho \pi \tau$ channel does not influence the results much. In this case some strength is missing at higher energies ($s \gtrsim 1.4$ GeV$^2$), but the peak definitely remains [25].

In principle, there are also final state interactions between the pions, which we assume to be negligible (see Fig. 4). One might argue that a considerable contribution from a $\sigma$ has been found in [3] and therefore this assumption is questionable. On the other hand in [12] the authors do not include an explicit $\sigma$ and they also successfully describe the data. Thus one encounters an uncertainty here, which, however, cannot be expected to influence the results qualitatively. The assumption that the pion correlations are negligible is supported by the observation that at tree level the direct three pion process (first two diagrams in Fig. 1) is much less important than the $\rho \pi \tau$ channel (third and fourth diagram in Fig. 1). In fact the success of the model also supports the assumption, although one might attribute small quantitative deviations to this approximation. One should recall that we only employ one parameter, which has to account for the position, the width and the strength of the peak. Thus, a semi-quantitative agreement with the data is already quite a success.

Another test of the model is given by using the same framework and parameters ($\mu_1 = M^2_\rho$ and $\mu_2 = 8.5 M^2_\rho$) to calculate the decay $\tau \rightarrow K \bar{K} \pi \nu_\tau$. In Fig. 5 we see the mass-squared distribution for the channel $\tau \rightarrow K \bar{K} \pi \nu_\tau$ in comparison to data from [28]. The ALEPH data include an axial and vector component whereas our calculation only considers the axial component. Therefore we also show the vector component as obtained from the BaBar $e^+e^-$ data. The ALEPH data have an axial and vector component whereas our calculation only considers the axial component. Therefore we also show the vector component as obtained from the BaBar $e^+e^-$ data. The results for the branching fractions are summarised in Table 1 and compared to the experimental data from [1] and [28] as well as to predictions from [30]. The result quoted in Table 1 for the total branching ratio in our model is obtained by using the calculated axial part and adding the experimental results for the vector part from [28] (taking the mean value and neglecting the uncertainty). We see that the axial fraction we get in our model

![Fig. 3. Spectral function for the decay $\tau \rightarrow 2\pi^{0}\pi^{+}\pi^{-}$ calculated by using different a subtraction point for the first loop ($\mu_2$) while keeping $\mu_1 = M^2_\rho$ in the scattering amplitude. Data are taken from [2].](image)

![Fig. 4. Diagram describing pion correlations, which we do not include in our calculation. The blob denotes the final state interactions of the pions.](image)

![Fig. 5. Mass-squared distribution for the predicted axial-vector component of the decay $\tau \rightarrow K \bar{K} \pi \nu_\tau$ for $\mu_1 = M^2_\rho$ and $\mu_2 = 8.5 M^2_\rho$, compared to vector plus axial-vector sum from [28]. The data points labelled ‘BaBar’ show the predictions for the vector component using the BaBar $e^+e^-$ data [28,29].](image)

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Branching ratio and axial fraction of the branching ratio for the decay $\tau \rightarrow K \bar{K} \pi \nu_\tau$ in comparison to data from [1] and [28] and the model from [30]. The vector part of the total branching ratio in our model is obtained by using the experimental data from [28].</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_{V+A}$</td>
<td>$B_{A}/B_{V+A}$</td>
</tr>
<tr>
<td>Our model</td>
<td>$5.5 \times 10^{-3}$</td>
</tr>
<tr>
<td>Model [30]</td>
<td>$5.6 \times 10^{-3}$</td>
</tr>
<tr>
<td>Exp.</td>
<td>$4.68 \pm 0.05 \times 10^{-3}$</td>
</tr>
</tbody>
</table>
is quite close to the experimental value, although one must add of course that the vector part has been obtained from the data. The bump in the ALEPH data is reproduced quite well and is mainly caused by phase space. Taking into account the vector component, we see that our results are little above the data. Nevertheless, the features are well described within the errorbars. The slight overlapping might be attributed to an overrating of $\mu_2$, but could as well be due to details of the spectral function of the $K^*$. However, keeping in mind that there is no free parameter for this comparison, the agreement is quite satisfying.

In a second calculation we explicitly introduced the $a_1$ in our calculation. In contrast to [12], where the width of the $a_1$ is parametrised, we generated the width by the decay of the $a_1$ into Goldstone bosons and vector mesons. In addition, we still include the WT term, since there is no reason to neglect it. A similar calculation without including the WT term has been done in [13] in the framework of the linear sigma model. We note that including the WT term and the elementary $a_1$ is not double counting at first place, since integrating out the $a_1$ would lead to a term of at least $O(q^4)$, whereas the WT term is $O(q^2)$. Nonetheless, since the integrated interaction has to be regularised, a partial double counting might enter through the renormalisation procedure [19].

Hence the WT term and its iteration should be taken into account also in the presence of a preformed $a_1$ state, but one has to be careful with the choice of the renormalisation point. This choice, however, can be compensated by adjusting the strength of the coupling of the $a_1$ to its decay channels.

The processes we include in the second calculation are all diagrams shown in Fig. 1. The scattering amplitude of the final state, represented by the blob, is different to the calculation before, since the kernel also contains the $a_1$ interaction (see Fig. 6). Microscopically such an explicit $a_1$ is a quark–antiquark state. On the hadronic level this intrinsic structure is not resolved and we treat the $a_1$ as an elementary field. The Lagrangian describing the decay of the $a_1$ into Goldstone boson and vector meson is given by

$$L_{AV\phi} = i c_1 Tr[V_{\mu\nu}^\dagger(A_{\mu}, u_1)] + i c_2 Tr[A_{\mu\nu}(V_{\mu}, u_1)],$$

where $V_{\mu}(A_{\mu})$ is the vector (axial-vector)-meson nonet, $V_{\mu\nu}(A_{\mu\nu})$ is the field strength of the vector (axial-vector) mesons and $u_\mu = i u D_\mu u$ with the usual non-linear representation of the pseudoscalar mesons $U$ and the covariant derivative $D_\mu$ [11]. For $c_1 = -\frac{1}{2}$ and $c_2 = -\frac{1}{2}$ Eq. (2) leads to the same expressions which have been found in [31,32]. We know already from the calculation before that a very small coupling of the $a_1$ and the parameters from the previous scenario would result in a good description. For an explicit $a_1$, however, we expect the size of the coupling to be around the values proposed in [31,32]. Of course, including the rescattering might change the values of $c_1$ and $c_2$, but changing them to 0 would mean one does not need an explicit $a_1$ (if there is no preformed component hidden in the renormalisation [19]). Therefore, we keep the contribution of the WT term small in order to be able to arrive at non-zero coupling constants $c_1$ and $c_2$. Most choices of the parameters lead to a double hump structure and certain parameter choices can also lead to a merging of these two bumps. These two possibilities are displayed in Fig. 7. The parameters for these curves are shown in Table 2. The merged bump can also qualitatively describe the data. However, the deviations in the low $s$ region, which could be seen already in the first calculation, are more pronounced in this case. On the other hand one might argue that the deviations on the right-hand side of the bump are less pronounced. However, we do not want to dwell upon the deviations to the data, but we want to emphasise the strong influence of the WT term. We note that by choosing the subtraction points according to Table 2, we already kept the contribution of the WT small (cf. Fig. 3). In order to generate a reasonable width of the $a_1$, the driving term should be the decay into vector meson and Goldstone boson. Since the WT term creates a peak by itself, that description leads to the appearance of a second peak, which might be interpreted as two independent sources of the $a_1$. Merging two bumps does not seem to be a natural way of describing the data. In other words: Why should an elementary state (i.e. a quark–antiquark state) appear right at the mass where an attractive potential (the WT interaction) has already created a resonant structure in the $\pi\rho$ system [23].

Finally, we want to remark that in a scenario without explicit $a_1$ it is possible to systematically improve the calculations by including higher order corrections to the WT term, which also reduces the dependence on the subtraction point $\mu_1$ [25]. Including the higher order terms, it is also possible to remove the kink seen at the threshold of the $KK^*$ in Fig. 3. In addition, these terms are sensible to the Dalitz projection data from [3], which is also addressed in [25].

To summarise, one finds that without the explicit $a_1$ one has a well behaved model, which describes the qualitative features of the data very well and even gives an approximately quantitative description, which can be systematically improved. All parameters, except for the subtraction points, are fixed by chiral symmetry breaking and the well-known properties of the $\rho$. Including an explicit $a_1$ in addition to the rescattering does not improve the

---

**Table 2**

<table>
<thead>
<tr>
<th>$M_{a_1}$ [GeV]</th>
<th>$f_\pi$</th>
<th>$c_1$</th>
<th>$c_2$</th>
<th>$\mu_{1}$ [GeV$^2$]</th>
<th>$\mu_{2}$ [GeV$^2$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Set 1</td>
<td>1.23</td>
<td>$\frac{f_\pi}{2M_\rho}$</td>
<td>$-\frac{1}{2}$</td>
<td>$-\frac{1}{2}$</td>
<td>$2M_\rho^2$</td>
</tr>
<tr>
<td>Set 2</td>
<td>1.21</td>
<td>$\frac{f_\pi}{2M_\rho}$</td>
<td>$-\frac{1}{2}$</td>
<td>$-\frac{1}{2}$</td>
<td>$M_\rho^2$</td>
</tr>
</tbody>
</table>

---

**Fig. 6.** The left-hand side shows the WT term alone, whereas the right-hand side shows in addition the $s$-channel diagram for an explicit $a_1$. In case the $a_1$ is included explicitly, we replace each dot in Fig. 2 by the sum of the WT term and the $s$-channel diagram.

**Fig. 7.** Spectral function for the decay $\tau \rightarrow 2\pi^0\rho$ including the $a_1$ with different sets of parameters in comparison to data from [2].
results and shows that the $a_1$ is strongly affected by the coupling to its decay channels. However, it is possible to get a qualitatively similar description of the data. Thus, one cannot rule out a remaining genuine quark component, but the indications point towards a substantial dynamically generated component.

Acknowledgements

We would like to thank M.F.M. Lutz and E. Kolomeitsev for useful discussions. We would also like to acknowledge stimulating discussions with E. Oset. We also thank Hasko Stenzel for clarifying our questions concerning the data and we are grateful to U. Mosel for stimulating discussions and continuous support.

References