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Kinetic theory for spin-polarized relativistic plasmas

Special Collection: Relativistic Plasma in Supercritical Electromagnetic Fields

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ABSTRACT

The investigation of spin and polarization effects in ultra-high intensity laser–plasma and laser–beam interactions has become an emergent topic in high-field science recently. In this paper, we derive a relativistic kinetic description of spin-polarized plasmas, where quantum-electrodynamics effects are taken into account via Boltzmann-type collision operators under the local constant field approximation. The emergence of anomalous precession is derived from one-loop self-energy contributions in a strong background field. We are interested, in particular, in the interplay between radiation reaction effects and the spin polarization of the radiating particles. For this, we derive equations for spin-polarized quantum radiation reaction from moments of the spin-polarized kinetic equations. By comparing with the classical theory, we identify and discuss the spin-dependent radiation reaction terms and radiative contributions to spin dynamics.

I. INTRODUCTION

There has recently been substantial recent interest in the effects of lepton and gamma-ray spin/polarization on strong-field quantum-electrodynamics (QED) emission processes that occur in ultraintense laser interactions with electron beams or plasma.[1–19] This is motivated by the continuing development of multi-petawatt class laser systems around the world, which should enable researchers to access QED critical strength fields in combination with relativistic plasma dynamics.[20] Strong field QED is important for particles with momentum p^μ interacting in electromagnetic fields for which the invariant quantity $\chi_p = ||F^{\mu\nu}p_\nu||/(mcE_S) \equiv [\gamma\sqrt{(\boldsymbol{E}+\boldsymbol{\beta}\times\boldsymbol{B})^2 - (\boldsymbol{\beta}\cdot\boldsymbol{E})^2}]/E_S$ is of order 1 or larger, where $E_S = m^2c^3/|e|\hbar$ is the Sauter–Schwinger field, often also called the QED critical field. The electromagnetic fields in the laboratory frame are typically much weaker than E_S. Under such conditions, the dominant quantum processes are the first order processes of photon emission by a lepton (nonlinear Compton scattering, in a laser field) or the decay of a photon into an electron–positron pair (the nonlinear Breit–Wheeler process). These processes depend on the spin/polarization state of both the leptons and photons.[1,2,8,21,22]

Studies of strong field QED have often made use of a semiclassical approach where the particles follow classical trajectories punctuated by point-like quantum emission/decay events that are calculated probabilistically under the assumptions that the “formation length” is short compared to the characteristic space/time scales of the electromagnetic fields (the local constant field approximation, LCFA); only first order processes are dominant; the fields are “weak” such that the field configuration in the rest frame of the particle is a crossed electric and magnetic field.[23–25] Motivated by this, recent studies have used an extended version of this model where the emission events are modified to include polarization dependent quantum rates combined with a classical spin-pusher using the Thomas Bargmann–Michel–Telegdi (T-BMT) equation.[4–11,13] However, this is done on an *ad hoc* basis, and it is not immediately clear that, for example, the inclusion of the anomalous $(g-2)$ magnetic moment in the T-BMT equation is consistent with the other assumptions. Other authors have been interested in the feedback of electron spin effects on classical expressions of radiation reaction.[26–29] In this paper, we develop a kinetic description for leptons in strong fields starting from transport equations resulting from a Wigner operator formalism in the mean field approximation[30] and include coupling of the fermion field to the high energy photons through a Boltzmann-like collision operator representing the point-like quantum emission/decay events.[31] By taking moments of a delta distribution in phase-space, we derive effective quasi-classical equations of motion for leptons, including the effects of spin and radiation reaction.

We start in Sec. II by reviewing the "classical" equations of motion with radiation reaction and note some properties, including the relative strength of spin corrections. In Sec. III, we derive the set of Boltzmann-like transport equations for the particles, comprising the classical transport and collision type operators for the quantum emissions. Section IV introduces the fluid equations for the transport by taking moments of the distribution functions.[32] In Sec. V, by assuming a delta distribution, we derive the single particle equations of motion and discuss the consequences of the expressions. Further discussion of the results is presented in Sec. VI. We conclude in Sec. VII. We employ rationalized Heaviside–Lorentz units with $\hbar = c = \varepsilon_0 = 1$ and Minkowski metric $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ throughout. We define dimensionless electron momenta according to $p^\mu/m \to p^\mu$ and coordinates according to $mx^\mu \to x^\mu$ and $m\tau \to \tau$ for proper time. We also absorb the magnitude of the electron charge $|e| = \sqrt{4\pi\alpha}$, where α is the fine structure constant, and the electron mass into the electromagnetic field strength according to $|e|F^{\mu\nu}/m^2 \to F^{\mu\nu}$. This means that in all the following equations, a factor of $q = \pm 1$ simply gives the sign of charge.

II. CLASSICAL RADIATION REACTION AND SPIN

A. Classical radiation reaction for orbital motion

It is a well known fact that the radiation emitted by accelerated charges acts back onto the electron motion.[38] For the classical orbital equations of motion, these so-called radiation reaction effects can be taken into account by adding a radiation reaction force f^μ_{rad} to the Lorentz force,

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = qF^{\mu\nu}u_\nu + f^\mu_{\mathrm{rad}}, \tag{1}$$

where τ is the particle's proper time, u^μ is the four-velocity/momentum, and $F^{\mu\nu}$ is the electromagnetic field tensor. Many forms of the radiation reaction force f^μ_{rad} have been proposed in the literature, but no definite consensus on the "correct" equation of motion has been reached so far. Let us write the radiation reaction force generically as follows: $f^\mu_{\mathrm{rad}} = \tau_R \Delta^{\mu\nu} R_\nu$, with the dimensionless radiation reaction time constant $\tau_R = 2\alpha/3$, and a projector $\Delta^{\mu\nu} = \eta^{\mu\nu} - u^\mu u^\nu$ onto the three-dimensional space-like hyperplane perpendicular to the four-velocity u^μ. This is necessary to ensure that the radiation reaction force and hence the total acceleration of the particle are orthogonal to its velocity, i.e., to ensure the validity of the subsidiary condition $u^2 = 1$.

The specific form of the vector R^μ varies for the different models of classical radiation reaction (RR). In particular, for the Lorentz–Abraham–Dirac (LAD) form of the RR force, we have[33,39,40]

$$R^\mu_{\mathrm{LAD}} = \ddot{u}^\mu, \tag{2}$$

such that $f^\mu_{\mathrm{rad}} = \tau_R(\ddot{u}^\mu + u^\mu \dot{u}^2)$ after an integration by parts.

However, the LAD form of radiation reaction is known to have some mathematical issues in the form of runaway solutions and pre-acceleration. These phenomena are related to the fact that LAD contains the second derivative of u, and a third initial condition—the initial acceleration—has to be provided. However, it is not independent, and only a proper choice leads to physical solutions of the LAD (see also detailed discussion in Refs. 41 and 42).

Many alternative forms of the equations of motion for classical radiation reaction have been derived in order to cure the deficiencies of the LAD equation. Most notably, the Landau–Lifshitz (LL) form of radiation reaction has been derived from LAD by reducing the order of the differential equation by iteration.[36] (See also the recent Refs. 43 and 44.) Iteration means treating the radiation reaction term in the LAD equation as a perturbation and approximating the jerk $\ddot{u}$ using the Lorentz force, $\ddot{u} = q\frac{\mathrm{d}(F\cdot u)}{\mathrm{d}\tau} = q(u.\partial F).u + qF.\dot{u}$. Iterating the LAD equation twice with this expression yields the LL equation, which has attracted some considerable interest in recent years due to its usefulness in numerical simulations of high-intensity laser–plasma interactions.

In fact, many of the RR models in the literature are connected to LAD by successive iterations and the quasi-constant approximations, even though their original derivation often was not following this path. Iterating LAD only once immediately leads to the Eliezer/Ford–O'Connell (EFO) equation,[34] and iterating a second time directly gives the Landau–Lifshitz (LL) form of RR as mentioned above. In addition to iterating the RR equations, one can also take a quasi-constant approximation by neglecting the $u.\partial F$ terms. The quasi-constant approximation of EFO is known as the Mo–Papas (MP) equation,[35] while the quasi-constant limit of LL is the Herrera (H) equation.[37] The specific forms of the radiation reaction vector R^μ and the RR force $\Delta^{\mu\nu}R_\nu$ are summarized in Table I. For a discussion of the different RR models in general, see Ref. 45, and as a limit of QED, see also Ref. 46.

B. Covariant form of spin precession with radiation reaction

The well established relativistic equation of motion for the covariant spin four-vector of a particle in a slowly varying external field is the Bargmann–Michel–Telegdi (BMT) equation,[47] which reads

TABLE I. Summary of radiation reaction models and their effect on spin precession.

	R^μ	$P^{\mu\nu}R_\nu$	$S.R$
LAD[33]	$\ddot{u}^\mu$	$\ddot{u}^\mu + u^\mu \dot{u}^2$	$S\cdot\ddot{u}$
EFO[34]	$q(u\cdot\partial)F^{\mu\nu}u_\nu + qF^{\mu\nu}\dot{u}_\nu$	$q(u\cdot\partial)F^{\mu\nu}u_\nu + qF^{\mu\nu}\dot{u}_\nu - qu^\mu u\cdot F\cdot\dot{u}$	$q(u\cdot\partial)S\cdot F\cdot u + qS\cdot F\cdot\dot{u}$
MP[35]	$qF^{\mu\nu}\dot{u}_\nu$	$qF^{\mu\nu}\dot{u}_\nu - qu^\mu u\cdot F\cdot\dot{u}$	$qS\cdot F\cdot\dot{u}$
LL[36]	$q(u\cdot\partial)F^{\mu\nu}u_\nu + F^{\mu\nu}F_{\nu\kappa}u^\kappa$	$q(u\cdot\partial)F^{\mu\nu}u_\nu + F^{\mu\nu}F_{\nu\kappa}u^\kappa - u^\mu\, u\cdot F^2\cdot u$	$q(u\cdot\partial)S\cdot F\cdot u + S\cdot F^2\cdot u$
H[37]	$F^{\mu\nu}F_{\nu\kappa}u^\kappa$	$F^{\mu\nu}F_{\nu\kappa}u^\kappa - u^\mu\, u\cdot F^2\cdot u$	$S\cdot F^2\cdot u$

$$\frac{dS^\mu}{d\tau} = q\frac{g}{2}\left[F^{\mu\beta}S_\beta + u^\mu(S_\alpha F^{\alpha\beta}u_\beta)\right] - u^\mu(S_\alpha \dot{u}^\alpha), \tag{3}$$

where S^μ is the spin four-vector, and g is the Landé factor. For an electron, $g \approx 2$, and the deviation from 2 is the anomalous magnetic moment $a_e \equiv (g-2)/2$. The one-loop perturbative QED result derived by Schwinger[48] is $a_e^{1-\text{loop}} = \alpha/2\pi$, and the experimentally measured value is $a_e = 1.15965218076(28) \times 10^{-3}$.[49]

The last term in Eq. (3) ensures that the spin four-vector S^μ remains orthogonal to the four-velocity u^μ. This is necessary since the spin-four vector is space-like, and in the electron, the rest frame has no time component. The form of Eq. (3) ensures that $S.u = 0$ as long as $u^2 = 1$, which then ensures that the length of the spin four-vector is conserved under the BMT equation $\frac{d(S\cdot S)}{d\tau} = 0$. This now has the consequence that the T-BMT equation also acquires a contribution from radiation reaction. Taking the form for the radiation reaction force as described above and plugging this into (3), we get

$$\frac{dS^\mu}{d\tau} = q\frac{g}{2}F^{\mu\nu}S_\nu - qa_e\, u^\mu(u\cdot F\cdot S) - \tau_R\, u^\mu; (S\cdot R), \tag{4}$$

where the last term appears due to the radiation reaction force. Clearly, different forms of R^μ for the various RR models will yield different contributions to the T-BMT equation, as shown in the last column of Table I.

Employing, for instance, the Herrera RR-force (LL in a quasi-constant field), we immediately get as the most simple consistent classical RR model with spin-precession,

$$\frac{du^\mu}{d\tau} = qF^{\mu\nu}u_\nu + \tau_R\left[F^{\mu\nu}F_{\nu\lambda}u^\lambda - u^\mu(u_\alpha F^{\alpha\beta}F_{\beta\lambda}u^\lambda)\right], \tag{5}$$

$$\frac{dS^\mu}{d\tau} = q\frac{g}{2}F^{\mu\nu}S_\nu - qa_e u^\mu\,(u_\alpha F^{\alpha\beta}S_\beta) - \tau_R\, u^\mu(S_\alpha F^{\alpha\beta}F_{\beta\lambda}u^\lambda). \tag{6}$$

The significance of the RR term in the second line of the T-BMT equation is to ensure orthogonality of the spin-vector and the velocity vector $S\cdot u = 0$ and the constancy of $S_\mu S^\mu$ for all times. It should be mentioned that these contributions are not small. If neglected, the solutions of the equations of motion vastly differ (see Fig. 1).

C. Noncovariant form of the T-BMT equation: Rest-frame spin vector

Here, we show that the RR term in the covariant T-BMT equation also gives a non-zero contribution to the precession of the spin vector in the rest frame of the particle and estimate the order of magnitude of the RR correction.

The equation of motion for spin vector $\boldsymbol{s}$ in the electron rest frame of the electron can be derived from the covariant BMT equation and has the following form:[38]

$$\frac{d\boldsymbol{s}}{d\tau} = q\boldsymbol{s}\times\Omega, \tag{7}$$

where

$$S^\mu = \left(\boldsymbol{u}\cdot\boldsymbol{s}, \boldsymbol{s} + \frac{\boldsymbol{u}(\boldsymbol{u}\cdot\boldsymbol{s})}{1+\gamma}\right), \quad \boldsymbol{s} = \boldsymbol{S} - \frac{\boldsymbol{u}(\boldsymbol{u}\cdot\boldsymbol{S})}{\gamma(1+\gamma)}. \tag{8}$$

The angular velocity vector Ω around which $\boldsymbol{s}$ precesses is derived straightforwardly from the covariant BMT equation (3), which first yields

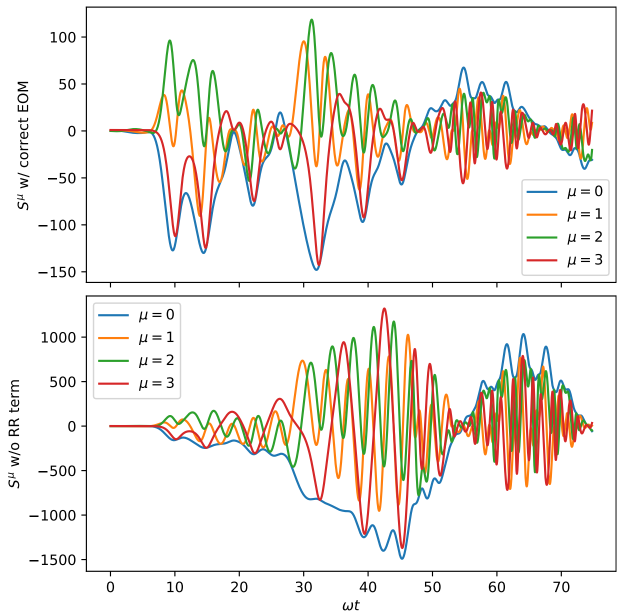


FIG. 1. Evolution of the spin-vector S^μ in a linearly polarized standing wave with $a_0 = 200$, obtained by numerically solving Eqs. (5) and (6). In the lower panel, the RR term in the BMT equation has been omitted, resulting in a completely wrong behavior of S^μ.

$$\frac{d\boldsymbol{s}}{d\tau} = \frac{qg}{2}\left[\boldsymbol{y} - \boldsymbol{u}\frac{y^0}{(1+\gamma)}\right] - \frac{\boldsymbol{s}\times(\boldsymbol{u}\times\dot{\boldsymbol{u}})}{1+\gamma}, \tag{9}$$

with $y^\mu = (y^0, \boldsymbol{y}) = F^{\mu\nu}S_\nu - u^\mu(u\cdot F\cdot S)$ and the last term being the Thomas precession.[38]

Without radiation reaction taken into account, $\dot{\boldsymbol{u}}$ is governed by the Lorentz force equation and the angular velocity vector is given by

$$\Omega = (1+a_e\gamma)\boldsymbol{B} - \left(a_e\gamma + \frac{\gamma}{1+\gamma}\right)(\boldsymbol{v}\times\boldsymbol{E}) - \frac{a_e\gamma^2}{1+\gamma}\boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{B}), \tag{10}$$

where $\boldsymbol{v} = \boldsymbol{u}/\gamma$. Here, it should be emphasized that all quantities defining Ω are taken in the laboratory frame, but $\boldsymbol{s}$ is the rest-frame spin vector.

With radiation reaction taken into account, we find that Ω acquires a contribution in addition to Eq. (10), due to the radiation reaction terms in the orbital equation Eq. (1),

$$\Omega \to \Omega + \delta\Omega_{RR}, \quad \delta\Omega_{RR} = \frac{\tau_R}{q}\frac{\gamma}{1+\gamma}(\boldsymbol{v}\times\boldsymbol{R}), \tag{11}$$

where $\boldsymbol{R}$ is the spatial part of R^μ. Clearly, the specific form of the radiation reaction correction to spin precession depends on the specific form of $\boldsymbol{R}$ in the various radiation reaction models. Taking here the most simple Herrera form, i.e., Landau–Lifshitz with negligible field gradients, $\boldsymbol{R}_H = \boldsymbol{L}\times\boldsymbol{B} + \boldsymbol{E}(\boldsymbol{E}\cdot\boldsymbol{u})$, with the Lorentz force vector $\boldsymbol{L} = \gamma\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}$, we find

$$\delta\Omega_{RR} = \frac{\tau_R\gamma^2}{q(1+\gamma)}[(\boldsymbol{v}\cdot\boldsymbol{B})(\boldsymbol{E}-\boldsymbol{v}\times\boldsymbol{B}) - (\boldsymbol{v}\cdot\boldsymbol{E})(\boldsymbol{B}-\boldsymbol{v}\times\boldsymbol{E})]. \tag{12}$$

Because of the cross product $\boldsymbol{v} \times \boldsymbol{R}_H$, only the sub-leading terms in the ultra-relativistic expansion of $\boldsymbol{R}_H$ contribute to $\delta\Omega_{\rm RR}$; the leading term drops out.

We can now estimate under what conditions the correction $\delta\Omega_{\rm RR}$ becomes relevant. For ultrarelativistic particles, $\gamma \gg 1$, the magnitude of $\delta\Omega_{\rm RR}$ can be estimated as $||\delta\Omega_{\rm RR}|| \sim \tau_R \gamma F^2$, where $F \sim ||\boldsymbol{E}||, ||\boldsymbol{B}||$ stands for the magnitude of the electromagnetic field in the laboratory frame. We have to compare it with the typical magnitude of the usual precession vector Ω, as shown in Eq. (10). The latter has two contributions, which scale as $||\Omega|| \sim F$ and $||\Omega|| \sim a_e \gamma F$ for the normal and anomalous precession, respectively. From this, we can derive two criteria, a classical one and a quantum one, under which the contribution $\delta\Omega_{\rm RR}$ becomes relevant.

First, the classical criterion compares the RR term with the normal precession. In order for the RR correction to become comparable, we have to fulfill the criterion $\tau_R \gamma F \sim 1$. With $\tau_R \sim \alpha$, this can be rewritten as $\gamma F / F_{\rm class,cr} \sim 1$, where $F_{\rm class,cr} = F_S/\alpha$ is the classical critical field, with the Schwinger field F_S.[50] Thus, according to this criterion, the RR corrections to the BMT equation become relevant if the background field in the rest frame of the particle is on the order of the classical critical field, which is much larger than the critical field of QED; hence, quantum effects can be expected to become important much earlier. This criterion can also be recast in the form $\alpha\chi \sim 1$, where χ is the quantum nonlinearity parameter (but $\alpha\chi$ is a classical parameter as $\hbar$ in α and χ cancel).

The second criterion is a quantum one and related to the anomalous magnetic moment. It reads $\tau_R F \sim a_e$. Taking for the anomalous moment, the 1-loop Schwinger value, $a_e = \alpha/2\pi$, we obtain that the corrections become relevant if $F/F_S \sim 1$. Thus, according to this criterion, radiation reaction effects for spin precession become relevant if the field in the laboratory frame is on the order of the Schwinger field.

Both of these corrections seem out of reach for the present day (and near future) high-intensity laser–plasma or laser–beam interactions. Those are typically characterized by $F \ll F_S$ and $\chi \gtrsim 1$. However, $\alpha\chi \sim 1$, as required by the first criterion, is still hard to reach. Moreover, for both criteria, the applicability of the classical approach ceases to be valid for much weaker fields than required for RR to affect spin precession. In particular, in the latter case, one already approaches the regime where one might see a breakdown of the Furry expansion of strong-field QED (Ritus–Narozhny conjecture).[51]

We thus can conclude that the kind of RR effects discussed so far is negligible from a phenomenological standpoint. Nonetheless, we should emphasize that RR effects are crucial in the covariant form of the T-BMT equation to ensure the orthogonality $u \cdot S = 0$ throughout the complete time evolution.

This concludes our classical investigations of the interplay between radiation reaction effects and spin precession. In the following, we will derive a kinetic description for polarized particles relevant for contemporary high-intensity laser–plasma interactions, taking into account quantum effects. From those, we will then derive effective semiclassical single-particle equations of motion for radiating particles with spin.

III. SPIN DEPENDENT KINETIC EQUATIONS

Here, we derive Boltzmann-type kinetic equations for describing high-intensity laser interactions with polarized particles. In principle, quantum transport theory, e.g., the Wigner operator formalism[30,52] or the nonequilibrium 2PI approach,[53] can provide equations for the evolution of statistical ensembles of particles to all orders in $\hbar$ and in the presence of strong background fields. However, the resulting equations are often extremely involved and thus require a number of approximations.

For high-intensity laser–plasma interactions, it was found[25,54] that typically the electromagnetic spectrum consists of two well-separated regions: (1) a coherent low-frequency peak describing the background/external and plasma fields including coherent radiation and (2) an incoherent high-frequency peak. This suggests that for the interaction of the fermions with a strong, weakly varying electromagnetic field one can perform a mean field (Hartree-type) approximation, in which fluctuations of the electromagnetic fields are neglected to lowest order and then introduced as a perturbation.

The quantum effects at $O(\hbar)$ in the interaction of the fermions with the strong mean-field background can typically be neglected if the field strength fulfills $F/\gamma F_S \ll 1$, i.e., the fields are weaker than γ times the Schwinger field, and the scale length of the background field $\ell = 1/||\nabla_x \ln F|| \gg \lambda\!\!\!^-_C$ is long compared to the Compton wavelength.[53,55] Both of these criteria are usually extremely well fulfilled for high-intensity laser–plasma interactions. Some phenomena occurring when the field strength is on the order of the Schwinger field have been discussed recently in Refs. 56 and 57.

The dominant quantum effects in laser–plasma interactions are hard photon emission by an electron and the decay of a photon into an electron positron pair. Those effects couple the fermionic degrees of freedom to the strong background field and (fluctuating) high-frequency field modes (photons), hence are beyond the mean field approximation.[53]

Our strategy, therefore, is as follows: We take the leading order of quantum transport theory following from the Wigner operator formalism to derive equations for the classical Vlasov-type transport of the particle number density and a spin density. Quantum effects will be included by constructing quantum collision operators with the spin-resolved photon emission (non-linear Compton) rates in the locally constant field approximation in their integral kernels. A rigorous derivation of the collisional quantum transport equations from first principles, but without special emphasis placed on the particle polarization, can be found, for instance, in a recent article by Fauth *et al.*[53]

A. Classical advection

Here, we derive the equations describing the classical transport of the on-shell particle density f and the spin-density a^μ in a strong background field from projections of the Wigner operator in the mean field approximation. Hence, we assume that the BBGKY hierarchy truncates at the one-body level, i.e., we ignore particle–particle correlations (lepton–lepton/lepton–ion collisions, etc.). This is valid for relatively dilute, high energy particle distributions. According to Vasak *et al.*,[30] the transport equations for the off-shell distributions $\mathcal{F}(x^\nu, p^\nu)$ and $\mathcal{A}^\mu(x^\nu, p^\nu)$, which are the scalar and axial-vector Fierz components of the Wigner function, read

$$p \cdot \mathcal{D}\mathcal{F} = 0, \tag{13}$$

$$p \cdot \mathcal{D}\mathcal{A}^\mu = -F^{\mu\nu}\mathcal{A}_\nu, \tag{14}$$

to lowest order in $\hbar$, complemented by the subsidiary condition $p_\mu \mathcal{A}^\mu = 0$, where p^ν is an off-shell momentum, $p^2 \neq 1$. At the same order, the operator $\mathcal{D}_\mu$ is given by

$$\mathcal{D}_\mu = \frac{\partial}{\partial x^\mu} + F_{\mu\nu}\frac{\partial}{\partial p_\nu}, \tag{15}$$

where $F_{\mu\nu}$ is the electromagnetic mean field. From quantum constraints at lowest order in $\hbar$, it follows that the distribution functions must be on-shell,[30] which we implement by writing

$$\mathcal{F}(x^\nu, p^\nu) = 2\delta(p^2-1)\theta(p^0)f(t,\boldsymbol{x},\boldsymbol{p}), \tag{16}$$
$$\mathcal{A}^\mu(x^\nu, p^\nu) = 2\delta(p^2-1)\theta(p^0)a^\mu(t,\boldsymbol{x},\boldsymbol{p}), \tag{17}$$

which singles out the positive energy (electron) mass shell. The transport equations for the on-shell distributions follow straightforwardly by integrating Eqs. (13) and (14) over dp^0. They read

$$\mathcal{T}f(t,\boldsymbol{x},\boldsymbol{p}) = 0, \tag{18}$$
$$\mathcal{T}a^\lambda(t,\boldsymbol{x},\boldsymbol{p}) = qF^{\lambda\nu}a_\nu\,. \tag{19}$$

From here on, the momentum p will always be on-shell with energy $\epsilon_{\boldsymbol{p}} = \sqrt{1+\boldsymbol{p}^2}$ and $p^2 = 1$. We also explicitly introduced the sign of the particle charge q, i.e., $q=-1$ for electrons. The corresponding transport equations for positrons can be obtained by setting $q=+1$.

The on-shell Vlasov-type transport operators on the left-hand side of Eqs. (18) and (19) are

$$\mathcal{T} = p\cdot\partial_x - qp\cdot F\cdot\partial_p = \epsilon_{\boldsymbol{p}}\partial_t + \boldsymbol{p}\cdot\nabla_{\boldsymbol{x}} + q\boldsymbol{L}\cdot\nabla_{\boldsymbol{p}}, \tag{20}$$

with the partial coordinate derivative $(\partial_x)_\mu = (\partial_t, \nabla_x)$, and the on-shell momentum partial derivative $\partial_p = (0, \nabla_{\boldsymbol{p}})$, and with $\boldsymbol{L} = \epsilon_{\boldsymbol{p}}\boldsymbol{E} + \boldsymbol{p}\times\boldsymbol{B}$.

Because of the subsidiary condition $p_\mu\mathcal{A}^\mu = 0$, the time component of the axial vector is not independent but rather a function of the spatial components. Therefore, similar to the treatment of the spin vector in Sec. II C, we can introduce an axial-vector (spin-)density $\boldsymbol{a}$ in the rest frame of the particle according to

$$a^\mu = \left(\boldsymbol{p}\cdot\boldsymbol{a}, \boldsymbol{a} + \frac{\boldsymbol{p}(\boldsymbol{p}\cdot\boldsymbol{a})}{1+\epsilon_{\boldsymbol{p}}}\right), \tag{21}$$

which fulfills the subsidiary condition $p\cdot a = 0$ automatically. Working here with $\boldsymbol{a}$ instead of the covariant a^μ will simplify the discussion spin-dependent radiation effects due to the RR term in the covariant classical BMT equation (6). The rest-frame spin density $\boldsymbol{a}$ obeys the following transport equation:

$$\mathcal{T}\boldsymbol{a} = q\boldsymbol{a}\times\left[\boldsymbol{B} - \frac{\boldsymbol{p}\times\boldsymbol{E}}{1+\epsilon_{\boldsymbol{p}}}\right]. \tag{22}$$

We can recognize the term in the square brackets on the right-hand side of the equation as Ω, Eq. (10), but without the anomalous magnetic moment, $a_e = 0$. Thus, so far the equations only describe the normal spin precession.

So far the transport equations are equivalent to purely classical descriptions and, at this level, do not mix f and $\boldsymbol{a}$. A mixture comes about through quantum effects. The quantum effects in the interaction of the electrons with the strong background fields have been neglected by going to the leading order in $\hbar$ in the quantum transport. The corresponding $\mathcal{O}(\hbar^1)$-terms have been calculated, e.g., in Ref. 55, and contain, for instance, spin-gradient forces. However, as we argued above that they are negligible for the typical conditions, we are interested in ($E/E_S \ll \epsilon_p, \ell \gg \lambda\!\!\!^-_C$). See also Ref. 16 for a detailed estimate of the size of spin-gradient forces.

What is *not* negligible, however, are the quantum effects due to a coupling of the charged particle dynamics in background fields to high-frequency photon modes. These interactions are the root cause of quantum radiation reaction and must be taken into account for a consistent description of high-intensity laser–plasma interactions. This will be done by adding to the right-hand side of Eqs. (18) and (19) appropriate collision operators. This approach is similar to what has been done previously in the literature,[31,58–60] where the kinetic equations were formulated for all particle polarization effects neglected. The details will be given in Sec. III B.

As has been discussed recently in the literature,[22,61,62] for a consistent treatment of strong-field QED effects, one needs to include also the effect of electron self-energy loop corrections at the same order in α. As will be discussed in more detail in Sec. III C, in the quantum kinetic approach, the only effect of the loop is to provide the anomalous spin-precession.

The full transport equations for a polarized plasma will, therefore, be of the following form:

$$\mathcal{T}f = \left(\frac{\partial f}{\partial\tau}\right)_{\text{rad}} + \left(\frac{\partial f}{\partial\tau}\right)_{\text{loop}}, \tag{23}$$

$$\mathcal{T}\boldsymbol{a} - q\boldsymbol{a}\times\Omega_0 = \left(\frac{\partial\boldsymbol{a}}{\partial\tau}\right)_{\text{rad}} + \left(\frac{\partial\boldsymbol{a}}{\partial\tau}\right)_{\text{loop}}, \tag{24}$$

where Ω_0 denotes Eq. (10) with $a_e = 0$.

B. Radiation emission contributions: Quantum collision operators

In this section, we are discussing how to include the effect of radiation emission in Eqs. (23) and (23). While the transport operators $\mathcal{T}$ describe the interaction of the electrons with the strong background fields in the mean field approximation, the radiation emission involves a coupling to high-energy photons, i.e., fluctuating field modes. To describe the photon emission process, we will employ the Furry expansion of strong-field QED at $O(\alpha)$.[51]

For ultrarelativistic particles $\epsilon_p \gg 1$, interaction with a strong field, the formation length of the photons is short compared to the scale lengths of the strong background field, and the photon emission rate can be calculated in the locally constant crossed field approximation (LCFA). The requirements are that the normalized scalar $(E^2 - B^2)$ and pseudoscalar $(\boldsymbol{E}\cdot\boldsymbol{B})$ field invariants are small compared to both unity and the quantum nonlinearity parameter $\chi \sim \epsilon_p E$. The formation length of high-energy photons becomes short if the parameter $\xi = E\ell \gg 1$, with the field scale length ℓ. For instance, in a plane wave, we might identify $\ell = \lambda\!\!\!^-_L$ with the reduced laser wavelength $\lambda\!\!\!^-_L = 1/\omega_L$ such that ξ becomes the usual classical laser-intensity parameter. In the quantum regime for $\chi \gtrsim 1$, it was found that additionally one has to require $\xi^3/\chi \gg 1$.[63] (See also Refs. 64–66 for further discussions of the limitations of the LCFA for soft photon emission.)

Under these conditions, photon emission is a *local* process (a short-range interaction) and can be described by local collision operators, $\mathcal{C}_f = (\frac{\partial f}{\partial\tau})_{\text{rad}}$ and $\mathcal{C}_{\boldsymbol{a}} = (\frac{\partial\boldsymbol{a}}{\partial\tau})_{\text{rad}}$, where each of the collision operators consists of the usual gain and loss terms, e.g., $\mathcal{C}_f = \dot{f}_{\text{gain}} - \dot{f}_{\text{loss}}$.

For ultrarelativistic particles, the photon is emitted into a narrow cone around its instantaneous velocity. We, therefore, can employ a collinear emission approximation, where the emitted photon momentum is assumed to be parallel to the electron momentum at the moment of emission. This assumption breaks the energy conservation during emission, but the energy lack is of order $1/\epsilon_p^2$, i.e., negligibly small for $\epsilon_p \gg 1$.[67]

The photon emission rates for polarized electrons can be represented using the Stokes vectors $\boldsymbol{n}_{i,f}$ of the incident/final electrons, which describe the state and degree of polarization of a particle. For instance, $\boldsymbol{n}_i^2 = 1$ means the incident particles are perfectly polarized and $\boldsymbol{n}_f^2 = 0$ means the final particles are completely unpolarized. In our kinetic approach, the role of the Stokes vectors is taken by the *local* polarization degree $\boldsymbol{s}(t,\boldsymbol{x},\boldsymbol{p}) \equiv \boldsymbol{a}(t,\boldsymbol{x},\boldsymbol{p})/f(t,\boldsymbol{x},\boldsymbol{p})$ of a plasma element in phase space.

The Stokes vectors, together with the LCFA building blocks for the differential rates w_A, where the label A runs over the various polarization contributions, completely describe the photon emission for polarized particles. The LCFA building blocks and the explicit formulas for the definition of the collinear differential photon emission rates $w_A(\boldsymbol{p},\boldsymbol{k})$ and the total photon emission rates $W_A(\boldsymbol{p})$ are given in Appendix A.

1. The gain term

The "gain" of the distribution functions at momentum $\boldsymbol{p}$ due to radiation emission must come from photon emission processes with *final* electron momentum $\boldsymbol{p}$, integrated over all possible initial states that lead to the said final state. If a photon with momentum $\boldsymbol{k}$ is emitted, the initial electron momentum was $\boldsymbol{p}+\boldsymbol{k}$ by means of the collinear emission approximation, integrated over all photon momenta $\boldsymbol{k}$. The integral kernels, thus, must be a linear combination of the distribution functions f, $\boldsymbol{a}$ at $\boldsymbol{p}+\boldsymbol{k}$, and the differential photon emission rates $w_A(\boldsymbol{p}+\boldsymbol{k},\boldsymbol{k})$.

The probability for photon emission with the initial and final electron polarization taken into account can be compactly expressed with help of the electron Stokes vectors and Müller matrices[22] as $R = \frac{1}{2}N_i M N_f^T$, where the Müller matrix collates the LCFA building blocks and $N_{i,f} = (1, \boldsymbol{n}_{i,f})$, see Appendix A. Calculating just the expression $N_i M$, with the identification of $\boldsymbol{n}_i \sim \boldsymbol{s}$, tell us the correct way how the LCFA building blocks have to be combined with the densities f and $\boldsymbol{a}$,

$$fN_iM = f(1,\boldsymbol{s})M \sim f\begin{pmatrix} w_0 + \boldsymbol{s}\cdot\boldsymbol{w}_i \\ \boldsymbol{w}_f + \boldsymbol{s}\cdot\underline{w}_{if}\end{pmatrix} \sim \begin{pmatrix}\dot{f}_{\text{gain}} \\ \dot{\boldsymbol{a}}_{\text{gain}}\end{pmatrix}. \tag{25}$$

Focusing for now on the gain term for f,

$$\dot{f}_{\text{gain}}(\boldsymbol{p}) = \int \frac{\mathrm{d}^3\boldsymbol{k}}{\omega_k}\mathcal{N} f(\boldsymbol{p}+\boldsymbol{k})[w_0(\boldsymbol{p}+\boldsymbol{k},\boldsymbol{k}) + \boldsymbol{s}(\boldsymbol{p}+\boldsymbol{k})\cdot\boldsymbol{w}_i(\boldsymbol{p}+\boldsymbol{k},\boldsymbol{k})]\,, \tag{26}$$

where ω_k is the photon frequency, and the normalization factor $\mathcal{N} = \epsilon_{\boldsymbol{p}}/\epsilon_{\boldsymbol{p}+\boldsymbol{k}}$ has to be included because the rates are expressed as probability per unit proper time for momentum $\boldsymbol{p}+\boldsymbol{k}$, while the final change on the left-hand side of Eq. (13) is with regard to momentum $\boldsymbol{p}$. The integral for $\dot{\boldsymbol{a}}_{\text{gain}}$ is constructed analogously.

2. The loss term

The loss terms describe the "loss" of particles from the momentum "mode" $\boldsymbol{p}$ due to radiation emission. Since any radiation emission alters the electrons' momentum state, the final electron polarization does not matter for the loss term and should be summed over. In order to find the form of the loss terms, it is useful to assume the case of electrons circulating a constant B-field. In this case, the direction $\hat{\mathcal{B}}$ is parallel to $\boldsymbol{B}$ in the laboratory frame, i.e., stationary. The particles will be polarized along this axis, and the polarization vector is nonprecessing.

Let us first introduce the fractions of particles f^σ with spin up ($\sigma = +1$) or down ($\sigma = -1$) as

$$f^\sigma = \frac{1}{2}(f + \sigma\,\hat{\mathcal{B}}\cdot\boldsymbol{a}) = \frac{1}{2}(f + \sigma a)\,. \tag{27}$$

In this representation, the loss terms for the $f^\sigma(\boldsymbol{p})$ due to photon emission, hence the electron losing momentum, can be straightforwardly written as

$$\dot{f}^\sigma_{\text{loss}}(\boldsymbol{p}) = f^\sigma(\boldsymbol{p})\sum_{\sigma'=\pm1} W^{\sigma\sigma'}(\boldsymbol{p}), \tag{28}$$

i.e., with the total emission rates summed over all possible spin-flip and non-flip transitions. Here, $W^{\sigma\sigma\prime} = \frac{1}{2}(W_0 + \sigma\,W_i + \sigma'\,W_f + \sigma\sigma'\,W_1)$; thus, $\sum_{\sigma'} W^{\sigma\sigma\prime} = (W_0 + \sigma W_i)$. With Eq. (27), it is now straightforward to read off the corresponding expressions for f and a as

$$\dot{f}_{\text{loss}} = fW_0 + aW_i, \tag{29}$$

$$\dot{a}_{\text{loss}} = aW_0 + fW_i\,. \tag{30}$$

This can now be generalized to arbitrary directions of $\boldsymbol{a}$ as

$$\dot{f}_{\text{loss}} = fW_0 + \boldsymbol{a}\cdot\boldsymbol{W}_i, \tag{31}$$

$$\dot{\boldsymbol{a}}_{\text{loss}} = \boldsymbol{a}W_0 + f;\boldsymbol{W}_i\,. \tag{32}$$

Combining everything from the preceding subsection, we obtain as results for the collision operators

$$\mathcal{C}_f(\boldsymbol{p}) = \int\frac{\mathrm{d}^3\boldsymbol{k}}{\omega_k}\frac{\epsilon_{\boldsymbol{p}}}{\epsilon_{\boldsymbol{p}+\boldsymbol{k}}}[f(\boldsymbol{p}+\boldsymbol{k})w_0(\boldsymbol{p}+\boldsymbol{k},\boldsymbol{k}) + \boldsymbol{a}(\boldsymbol{p}+\boldsymbol{k})\cdot\boldsymbol{w}_i(\boldsymbol{p}+\boldsymbol{k},\boldsymbol{k})] - f(\boldsymbol{p})W_0(\boldsymbol{p}) - \boldsymbol{a}(\boldsymbol{p})\cdot\boldsymbol{W}_i(\boldsymbol{p}), \tag{33}$$

$$\mathcal{C}_{\boldsymbol{a}}(\boldsymbol{p}) = \int\frac{\mathrm{d}^3\boldsymbol{k}}{\omega_k}\frac{\epsilon_{\boldsymbol{p}}}{\epsilon_{\boldsymbol{p}+\boldsymbol{k}}}[f(\boldsymbol{p}+\boldsymbol{k})\boldsymbol{w}_f(\boldsymbol{p}+\boldsymbol{k},\boldsymbol{k}) + \boldsymbol{a}(\boldsymbol{p}+\boldsymbol{k})\cdot\underline{w}_{if}(\boldsymbol{p}+\boldsymbol{k},\boldsymbol{k})] - f(\boldsymbol{p})\boldsymbol{W}_i(\boldsymbol{p}) - \boldsymbol{a}(\boldsymbol{p})W_0(\boldsymbol{p})\,. \tag{34}$$

C. Electron self-energy loop contributions and anomalous precession

As has been discussed recently in the literature, the inclusion of loop contributions at the same order in α as the emission processes is necessary to maintain unitarity.[22,61,62] For high-energy photon emission at order α, the corresponding loop contribution originates from the interference of the electron self-energy loop with the $\mathcal{O}(\alpha^0)$ part of electron propagation. As has been argued, in, e.g.,[22] a convenient way of implementing this can be achieved by taking the combined Müller matrices for emission, M, and the loop M^L, $M + M^L$ especially in the resummation of quantum radiation reaction effects.[29] We have to adapt this to our kinetic description.

The basic strategy for constructing the loop contribution to the kinetic equations will be as follows: Calculate for the loop contribution, e.g., $(\partial f/\partial\tau)_{\text{loop}}$, the same combinations of the LCFA building blocks as for the emission operator, just with the corresponding expressions replaced by the loop expressions $R_A \to R_A^L$ [see Eqs. (A16)–(A18)]. The loop insertion does not change the particle momentum, and the expressions R_A^L given in Appendix A are already integrated over all photon momenta running around the loop. Thus, in the gain-parts of $(\partial f/\partial\tau)_{\text{loop}}$ and $(\partial \boldsymbol{a}/\partial\tau)_{\text{loop}}$, we actually do not have any expressions, which have to be integrated over photon momenta.

For the scalar density f, this means that the loop contribution must be zero, and indeed,

$$\left(\frac{\partial f}{\partial\tau}\right)_{\text{loop}} = fR_0^L + \boldsymbol{a}\cdot\boldsymbol{R}_i^L - fR_0^L - \boldsymbol{a}\cdot\boldsymbol{R}_i^L = 0\,. \tag{35}$$

For the axial-vector density, we obtain the nonzero result

$$\begin{aligned}\left(\frac{\partial \boldsymbol{a}}{\partial\tau}\right)_{\text{loop}} &= f\boldsymbol{R}_f^L + \boldsymbol{a}\cdot \underline{R}_{if}^L - f\boldsymbol{R}_i^L - \boldsymbol{a}R_0^L \\ &= \alpha\int_0^1 \mathrm{d}\lambda\lambda\frac{\mathrm{Gi}(z)}{\sqrt{z}}\tilde{\boldsymbol{a}}\cdot(\hat{\mathcal{K}}\hat{\mathcal{E}} - \hat{\mathcal{E}}\hat{\mathcal{K}})\,,\end{aligned} \tag{36}$$

where $\hat{\mathcal{E}}$, $\hat{\mathcal{K}}$, and $\hat{\mathcal{B}}$ are unit vectors related to the instantaneous rest frame field components and form an approximately orthogonal basis, see Appendix A. This is nothing but the anomalous spin-precession in disguise. To see this clearly, we first note that the integral over the Scorer function Gi yields the one-loop field-dependent value of the anomalous magnetic moment,

$$a_e(\chi) = \frac{\alpha}{\chi}\int_0^1 \mathrm{d}\lambda\lambda\frac{\mathrm{Gi}(z)}{\sqrt{z}}\,, \tag{37}$$

which was derived by Ritus.[68] For weak fields, $\chi \to 0$, it approaches Schwinger's 1-loop value $a_e(\chi\to 0) = \alpha/2\pi$, and $a_e(\chi)$ decreases monotonically to zero as χ increases. We, thus, have for the loop contribution to the collision operator for the axial vector the following result:

$$\begin{aligned}\left(\frac{\partial \boldsymbol{a}}{\partial\tau}\right)_{\text{loop}} &= \chi a_e(\chi)\boldsymbol{a}\cdot(\hat{\mathcal{K}}\hat{\mathcal{E}} - \hat{\mathcal{E}}\hat{\mathcal{K}}) \\ &= -\chi a_e(\chi)\boldsymbol{a}\times\hat{\mathcal{B}} = -\boldsymbol{a}\times\boldsymbol{\Omega}_{\text{anomalous}}(a_e(\chi))\,,\end{aligned} \tag{38}$$

ignoring a term in $(\hat{\mathcal{E}}\cdot\hat{\mathcal{B}})\hat{\mathcal{E}}$, which should be negligible under the LCFA approximation and with the anomalous part of the spin-precession vector, as shown in Eq. (10), $\boldsymbol{\Omega}_{\text{anomalous}} = \boldsymbol{\Omega} - \boldsymbol{\Omega}_0 = a_e\chi\hat{\mathcal{B}}$.

In summary, we have shown here that in the kinetic approach, the loop contribution gives exactly the anomalous term of spin-precession, with the field-dependent anomalous moment $a_e(\chi)$. Combining this with the normal precession, which was obtained from the classical transport, it becomes clear that the axial-vector density $\boldsymbol{a}(t,\boldsymbol{x},\boldsymbol{p})$ at each point in phase space precesses around a local value of Ω as given in Eq. (10), but with the field-dependent anomalous moment $a_e \to a_e(\chi)$.

IV. MOMENT HIERARCHY AND RELATIVISTIC FLUID EQUATIONS

An infinite chain of equations describing the transport of bulk plasma properties, including particle number densities, energy density, pressure, heat, etc., can be obtained by calculating momentum moments of the kinetic equations[69] and truncated by some choice of closure to yield fluid equations for the electron and positron species in the plasma.

Here, we derive two-fluid equations for the electron and positron components of a relativistic plasma[70] with the particle spin-polarization taken into account up to second order. Our aim will be, in particular, to work out the effects of the spin dependence in the collision operators and how they affect radiation reaction. The moment equations for a nonrelativistic electron gas with spin effects were discussed in Ref. 71. A relativistic transport equation for particles with spin was given in[72] using an extended phase space, see also Refs. 73–75 and 71, for the equivalence of the extended phase space with our method.

In order to calculate the relativistic moment equations, we have to integrate the two kinetic equations for f and $\boldsymbol{a}$ for a generic moment of the kernel $\Psi = \Psi(p^\alpha, p^\beta, \ldots)$ and with the Lorentz invariant measure $\int \mathrm{d}^3\boldsymbol{p}/\epsilon_p$. Introducing the short-hand notation for the momentum space integrals $\langle Y\rangle = \int\frac{\mathrm{d}^3\boldsymbol{p}}{\epsilon_p} fY$ with f as the weight, we find

$$\frac{\partial}{\partial x^\mu}\langle\Psi p^\mu\rangle = -qF_\mu^{\ \nu}\left\langle p^\mu\frac{\partial\Psi}{\partial p^\nu}\right\rangle + \int\frac{\mathrm{d}^3\boldsymbol{p}}{\epsilon_p}\Psi\mathcal{C}_f, \tag{39}$$

$$\frac{\partial}{\partial x^\mu}\langle\Psi p^\mu\boldsymbol{s}\rangle = -qF_\mu^{\ \nu}\left\langle \boldsymbol{s}p^\mu\frac{\partial\Psi}{\partial p^\nu}\right\rangle + q\langle\Psi\boldsymbol{s}\times\Omega\rangle + \int\frac{\mathrm{d}^3\boldsymbol{p}}{\epsilon_p}\Psi\mathcal{C}_{\boldsymbol{a}}\,, \tag{40}$$

where in terms involving the electromagnetic fields we have integrated by parts and neglected the surface contributions. We recall the definition of polarization degree $\boldsymbol{s}\equiv\boldsymbol{a}/f$.

A. Moments of the scalar distribution f

1. Zero order moment

For the zero order moment, i.e., with the kernel function $\Psi = 1$, of the scalar transport equation for f, we find number density current conservation

$$\frac{\partial}{\partial x^\mu}J^\mu = \int\frac{\mathrm{d}^3\boldsymbol{p}}{\epsilon_p}\Psi\mathcal{C}_f = 0, \tag{41}$$

where the current density is

$$J^\mu = \int\frac{\mathrm{d}^3\boldsymbol{p}}{\epsilon_p}p^\mu f = \langle p^\mu\rangle\,. \tag{42}$$

Clearly, the derivative in the field advection term in Eq. (39) vanishes since Ψ is a constant. The integral over the collision operator $\mathcal{C}_f$ vanishes because the gain and loss terms must exactly cancel when integrated over all electron momenta $\boldsymbol{p}$ to conserve the electron number during photon emission. To see this explicitly, one may conveniently substitute the integration variable in the gain term $\boldsymbol{p} = \boldsymbol{q} - \boldsymbol{k}$, with $\mathrm{d}^3\boldsymbol{p} = \mathrm{d}^3\boldsymbol{q}$ to arrive at

$$\int \frac{d^3 \boldsymbol{p}}{\epsilon_p} \mathcal{C}_f^{\text{gain}}(\boldsymbol{p}) = \int \frac{d^3 \boldsymbol{q}}{\epsilon_q} \int \frac{d^3 \boldsymbol{k}}{\omega_k} [f(\boldsymbol{q}) w_0(\boldsymbol{q}, \boldsymbol{k}) + \boldsymbol{a}(\boldsymbol{q}) \cdot \boldsymbol{w}_i(\boldsymbol{q}, \boldsymbol{k})]$$
$$= \int \frac{d^3 \boldsymbol{q}}{\epsilon_q} f(\boldsymbol{q}) \int \frac{d^3 \boldsymbol{k}}{\omega_k} w_0(\boldsymbol{q}, \boldsymbol{k}) + \int \frac{d^3 \boldsymbol{q}}{\epsilon_q} \boldsymbol{a}(\boldsymbol{q})$$
$$\cdot \int \frac{d^3 \boldsymbol{k}}{\omega_k} \boldsymbol{w}_i(\boldsymbol{q}, \boldsymbol{k}) = \int \frac{d^3 \boldsymbol{q}}{\epsilon_q} \mathcal{C}_f^{\text{loss}}(\boldsymbol{q}). \quad (43)$$

2. First order moment

Next, we discuss the first order moment of f, with $\Psi = p^\mu$. The left-hand side of the equation can be rewritten as the divergence of the energy-momentum tensor of the plasma, $T^{\mu\nu} = \langle p^\mu p^\nu \rangle$. Evaluating the momentum derivatives in the field-advection term, it takes the form of the Lorentz force. Combining with the collision terms, we obtain

$$\partial_\mu T^{\mu\nu} = qF^{\nu\mu} J_\mu - \langle p^\nu (I_0 + \boldsymbol{s} \cdot \boldsymbol{I}_i) \rangle, \quad (44)$$

where $I_A = \int d\lambda \lambda \frac{dR_A}{d\lambda}$ are the first moments of the photon spectrum, i.e., the normalized radiated power.

We now need to show that the integral over the collision operator $\mathcal{C}_f$ with $\Psi = p^\mu$ yields the last term in (44). To see this, we first make the same substitution $\boldsymbol{p} = \boldsymbol{q} - \boldsymbol{k}$ as in (43). Considering energy conservation, we write $\epsilon_{\boldsymbol{p}} = \epsilon_{\boldsymbol{q}} - \omega_k + \delta$, where δ is the energy error introduced by not including the momentum contribution from the background fields.[76] For ultrarelativistic particles and in the collinear emission approximation,[23] we have $\delta = 1/2|\boldsymbol{q}| - 1/2|\boldsymbol{p}| \approx \frac{\lambda}{2\epsilon_p(1-\lambda)}$. The corresponding term of the integrated collision operator is by a factor of $1/\epsilon_p^2 \ll 1$ smaller than the leading order term and can, therefore, be neglected. The leading order term reads

$$\int \frac{d^3 \boldsymbol{p}}{\epsilon_p} p^\mu \mathcal{C}_f(\boldsymbol{p}) = -\int \frac{d^3 \boldsymbol{p}}{\epsilon_p} \int \frac{d^3 \boldsymbol{k}}{\omega_k} k^\mu [f(\boldsymbol{p}) w_0(\boldsymbol{p}, \boldsymbol{k}) + \boldsymbol{a}(\boldsymbol{p}) \cdot \boldsymbol{w}_i(\boldsymbol{p}, \boldsymbol{k})]. \quad (45)$$

For the integral over the photon momentum, we make use of the definition of the rate w_A, which is given in Appendix A, Eq. (A1). By means of the delta function, the photon four-momentum $k^\mu = (|\boldsymbol{k}|, \boldsymbol{k}) \to \lambda(|\boldsymbol{p}|, \boldsymbol{p}) \simeq \lambda p^\mu$, and thus,

$$\int \frac{d^3 \boldsymbol{p}}{\epsilon_p} p^\mu \mathcal{C}_f(\boldsymbol{p}) = -\int \frac{d^3 \boldsymbol{p}}{\epsilon_p} p^\mu [f(\boldsymbol{p}) I_0 + \boldsymbol{a}(\boldsymbol{p}) \cdot \boldsymbol{I}_i]$$
$$= -\langle p^\mu (I_0 + \boldsymbol{s}(\boldsymbol{p}) \cdot \boldsymbol{I}_i) \rangle. \quad (46)$$

This represents the (momentum averaged) effect of quantum radiation reaction due to hard photon emission. In particular, the term $\boldsymbol{s}(\boldsymbol{p}) \cdot \boldsymbol{I}_i \sim s_B I_i$ describes the spin-polarization dependence of radiative energy loss.

3. Second-order moment

For the second-order moment with weight $\Psi = p^\nu p^\lambda$, we led to the definition of the stress-flow tensor[77] $U^{\mu\nu\lambda} = \langle p^\mu p^\nu p^\lambda \rangle$, the trace of which equals the current $U^{\mu\nu}_\mu = J^\nu$, and which obeys the following transport equation:

$$\partial_\mu U^{\mu\nu\lambda} = qF^{\nu\mu} T^\lambda_\mu + qF^{\lambda\mu} T^\nu_\mu + \langle p^\nu p^\lambda (K_0 + \boldsymbol{s} \cdot \boldsymbol{K}_i) \rangle$$
$$-2\langle p^\nu p^\lambda (I_0 + \boldsymbol{s} \cdot \boldsymbol{I}_i) \rangle. \quad (47)$$

The quantities $K_A = \int_0^1 d\lambda \lambda^2 dR_A/d\lambda$ are the second moments of the photon emission spectrum and describe the electron momentum spreading due to the stochasticity of quantized photon emission. The term in the second line of Eq. (47) is the spin-dependent radiative cooling effect.

B. Moments of the axial-vector equation $\boldsymbol{a}$

1. Zero-order moment

Working out the zeroth moment, $\Psi = 1$, of the axial vector equation, we derive an equation for the spin current,

$$\boldsymbol{J}_s^\mu = \int \frac{d^3 \boldsymbol{p}}{\epsilon_p} p^\mu \boldsymbol{a}(\boldsymbol{p}) = \langle p^\mu \boldsymbol{s} \rangle. \quad (48)$$

As for the scalar current, the field advection term vanishes. However, the spin current is not conserved. It obeys the following transport equation:

$$\partial_\mu \boldsymbol{J}_s^\mu = q\langle \boldsymbol{s} \times \Omega \rangle + \langle \boldsymbol{W}_f - \boldsymbol{W}_i + \boldsymbol{s} \cdot \underline{W}_{if} - \boldsymbol{s} W_0 \rangle. \quad (49)$$

The first term on the right-hand side follows straightforwardly from spin-precession term in the transport equation [Eq. (24)]. It causes a depolarization of the plasma if $\boldsymbol{s}(t, \boldsymbol{x}, \boldsymbol{p})$ precesses at different frequency at either different points in space (i.e., different field strength) or for different momenta.[16,78] The second term stems from integral over the collision operator $\mathcal{C}_{\boldsymbol{a}}$, where the calculation proceeds similar to the scalar case $\mathcal{C}_f$ above. This second term is responsible for the radiative polarization of the electrons, including the Sokolov–Ternov[79] or Baier–Katkov–Strakhovenko[80,81] radiative polarization effects.

With help of the expressions given in Appendix A, the combination of rates in the radiative polarization term can be reformulated as

$$\boldsymbol{W}_f - \boldsymbol{W}_i + \boldsymbol{s} \cdot \underline{W}_{if} - \boldsymbol{s} W_0$$
$$= \hat{\mathcal{B}}(W_f - W_i) + (s_B \hat{\mathcal{B}} + s_E \hat{\mathcal{E}}) W_3 + s_K \hat{\mathcal{K}} W_4$$
$$= \alpha \int_0^1 d\lambda \frac{\lambda^2}{1-\lambda} \left[\boldsymbol{s} \frac{\text{Ai}'(z)}{z} - \hat{\mathcal{B}} \frac{\text{Ai}(z)}{\sqrt{z}} - s_K \hat{\mathcal{K}} \left(\text{Ai}_1(z) + \frac{\text{Ai}'(z)}{z} \right) \right], \quad (50)$$

where we have expanded the spin polarization $\boldsymbol{s}$ in the three principal directions along the electric (magnetic) field $\hat{\mathcal{E}}$ ($\hat{\mathcal{B}}$) in the particle rest frame, and $\hat{\mathcal{K}} = \hat{\mathcal{E}} \times \hat{\mathcal{B}}$, according to $\boldsymbol{s} = s_B \hat{\mathcal{B}} + s_E \hat{\mathcal{E}} + s_K \hat{\mathcal{K}}$.

2. Higher moments

With the weight functions $\Psi = p^\nu$ and $\Psi = p^\nu p^\lambda$, we define the spin energy-momentum tensor $\boldsymbol{T}_s^{\mu\nu} = \langle p^\mu p^\nu \boldsymbol{s} \rangle$ and spin stress-flow tensor $\boldsymbol{U}_s^{\mu\nu\lambda} = \langle p^\mu p^\nu p^\lambda \boldsymbol{s} \rangle$, respectively. They obey the following transport equations:

$$\partial_\mu \boldsymbol{T}_s^{\mu\nu} = qF^{\nu\mu} (\boldsymbol{J}_s)_\mu + q\langle p^\nu \boldsymbol{s} \times \Omega \rangle$$
$$+ \langle p^\nu (\boldsymbol{W}_f - \boldsymbol{W}_i + \boldsymbol{s} \cdot \underline{W}_{if} - \boldsymbol{s} W_0) \rangle$$
$$- \langle p^\nu (\boldsymbol{I}_f + \boldsymbol{s} \cdot \underline{I}_{if}) \rangle, \quad (51)$$

$$\partial_\mu \boldsymbol{U}_s^{\mu\nu\lambda} = qF^{\nu\mu}(\boldsymbol{T}_s)^{\lambda}_{\mu} + qF^{\lambda\mu}(\boldsymbol{T}_s)^{\nu}_{\mu} + q\langle p^\nu p^\lambda \boldsymbol{s}\times\Omega\rangle + \langle p^\nu p^\lambda(\boldsymbol{W}_f - \boldsymbol{W}_i + \boldsymbol{s}\cdot \underline{W}_{if} - \boldsymbol{s}W_0)\rangle - 2\langle p^\nu p^\lambda(\boldsymbol{I}_f + \boldsymbol{s}\cdot \underline{I}_{if})\rangle + \langle p^\nu p^\lambda(\boldsymbol{K}_f + \boldsymbol{s}\cdot \underline{K}_{if})\rangle. \tag{52}$$

V. SINGLE PARTICLE EQUATIONS OF MOTION

From the moment hierarchy, we can now derive effective semi-classical single-particle equations of motion for a radiating lepton including the effects of its spin polarization. Note that the spin polarization is a classical quantity representing the expectation of the four components of the particle spin vector and so the single particle equations should be interpreted as the trajectory of an "average" lepton and not of an individual lepton. To find those, we first note that the current density for a point particle may be written as[82]

$$J^\mu(x) = \int \mathrm{d}\tau\, u^\mu(\tau)\delta^{(4)}(x - x(\tau))\,. \tag{53}$$

By analogy with Eq. (42), using the definition in Eq. (48), we deduce that the appropriate form of the spin current density for a point particle should be

$$J_s^\mu(x) = \int \mathrm{d}\tau\, u^\mu(\tau)\boldsymbol{s}(\tau)\delta^{(4)}(x - x(\tau))\,. \tag{54}$$

Hence, the scalar and axial vector distribution functions for a single point particle are

$$f(t,\boldsymbol{x},\boldsymbol{p}) = \int \mathrm{d}\tau\epsilon_{\boldsymbol{p}}\delta^{(3)}(\boldsymbol{p} - \boldsymbol{u}(\tau))\delta^{(4)}(x - x(\tau)), \tag{55}$$

$$\boldsymbol{a}(t,\boldsymbol{x},\boldsymbol{p}) = \int \mathrm{d}\tau\boldsymbol{s}(\tau)\epsilon_{\boldsymbol{p}}\delta^{(3)}(\boldsymbol{p} - \boldsymbol{u}(\tau))\delta^{(4)}(x - x(\tau))\,. \tag{56}$$

From these definitions of the distribution functions, the energy momentum tensor for a point particle is

$$T^{\mu\nu} = \int \mathrm{d}\tau;\, u^\mu(\tau)u^\nu(\tau)\delta^4(x - x(\tau))\,. \tag{57}$$

From this, we can derive the momentum conservation equation, starting with

$$\begin{aligned}\partial_\mu T^{\mu\nu} &= \int \mathrm{d}\tau u^\nu(\tau)u^\mu(\tau)\partial_\mu\delta^4(x - x(\tau)) \\ &= -\int \mathrm{d}\tau u^\nu(\tau)\frac{d}{d\tau}\delta^4(x - x(\tau)) \\ &= \int \mathrm{d}\tau\frac{\mathrm{d}u^\nu(\tau)}{\mathrm{d}\tau}\delta^4(x - x(\tau)).\end{aligned} \tag{58}$$

Thus, from Eq. (44), we find

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = qF^{\mu\nu}(x(\tau))u_\nu(\tau) - u^\mu(\tau)(I_0(\chi) + \boldsymbol{s}(\tau)\cdot \boldsymbol{I}_i(\chi)), \tag{59}$$

where $\chi = \chi(x(\tau), u(\tau))$ is the local value of the quantum parameter at the particle location. This quasi-classical equation for the particle acceleration contains the Lorentz force and additional a spin-dependent radiation reaction term as the second term on the right. In the limit of unpolarized particles $\boldsymbol{s} = 0$, our result agrees with the literature, e.g., Refs. 32, 58, and 83.

The quasi-classical equation for the polarization vector evolution is obtained from Eq. (48) as

$$\frac{\mathrm{d}\boldsymbol{s}}{\mathrm{d}\tau} = q\boldsymbol{s}(\tau)\times\Omega(\tau) + \boldsymbol{W}_f - \boldsymbol{W}_i + \boldsymbol{s}\cdot(\underline{W}_{if} - W_0\mathbf{1}), \tag{60}$$

where again the four-divergence is transformed into a proper time derivative. Equation (60) describes the spin precession, here with the anomalous magnetic moment being the field-dependent value $a_e(\chi)$, as shown in Eq. (37). The additional terms describe the radiative polarization of the electrons, hence are the generalization of the Sokolov–Ternov effect. In the limit of weak fields, i.e., in the lowest order in χ, this equation is known as the Baier–Katkov–Strakhovenko equation.[80,81] Both these effects have been recently employed for simulating spin effects in laser–electron interactions, see, e.g., Refs. 13, 15, and 84. While a derivation of the precession part had been given recently in Refs. 22 and 85, no formal derivation of the full equation is known to us.

VI. DISCUSSION

A. Comparison of the kinetic equation results with classical RR and the spin-dependent Gaunt factor

By comparing the kinetic equation results for the radiative energy loss with the classical RR equations, we can subsume all quantum effects into a spin-dependent Gaunt factor $\mathfrak{g}_s$,

$$\langle p^\mu(I_0 + \boldsymbol{s}\cdot\boldsymbol{I}_i)\rangle = \langle p^\mu \mathfrak{g}_s(\chi)I_{\text{class}}\rangle, \tag{61}$$

where $I_{\text{class}} = \frac{2}{3}\alpha m\chi^2$ is the lowest order term in the asymptotic expansion of I_0 as $\chi\to 0$. Thus,

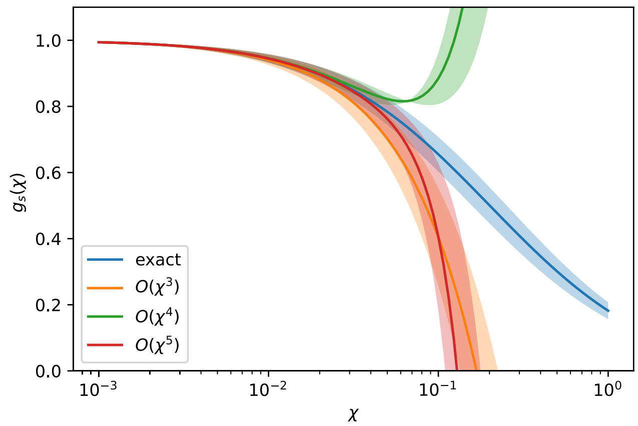
$$\mathfrak{g}_s(\chi) = -\frac{3}{2\chi^2}\int_0^1 \mathrm{d}\lambda\lambda\left(\mathrm{Ai}_1(z) + 2g\frac{\mathrm{Ai}'(z)}{z} + \lambda\frac{\mathrm{Ai}(z)}{\sqrt{z}}(\boldsymbol{s}\cdot\hat{\mathcal{B}})\right). \tag{62}$$

By using the tables in Appendix B, we can write down the series expansion of the spin-dependent Gaunt factor for small $\chi \ll 1$ as

$$\mathfrak{g}_s(\chi) \simeq 1 - \left(\frac{55\sqrt{3}}{16} + \frac{3}{2}s_B\right)\chi + \left(48 + \frac{105\sqrt{3}}{8}s_B\right)\chi^2 - \left(\frac{8855\sqrt{3}}{32} + 300s_B\right)\chi^3. \tag{63}$$

As can be seen from this equation, only the degree of polarization along the rest frame magnetic field, $s_B = \boldsymbol{s}\cdot\hat{\mathcal{B}}$, affects the Gaunt factor. Positive values of s_B reduce the Gaunt factor and hence radiative losses, while negative s_B increases $\mathfrak{g}_s$. Thus, in an arbitrary electromagnetic field, electrons that have some degree of polarization in the direction of the rest frame magnetic field experience less radiation reaction than unpolarized electrons. The polarization orthogonal to $\hat{\mathcal{B}}$ does not affect the strength of radiative losses. The function $\mathfrak{g}_s$ is plotted in the upper panel of Fig. 2

By inspecting the effective single-particle equation, Eq. (59), it is straightforward to find that (59) violates the relativistic constraint $\dot{u}\cdot u = 0$.[32,58,83] This behavior is a consequence of the ultra-relativistic and collinear emission approximation made in the collision operators, under which we kept only the leading order terms in inverse power of γ. It can be seen that in the limit $\mathfrak{g}_s\to 1$, Eq. (59) agrees with the



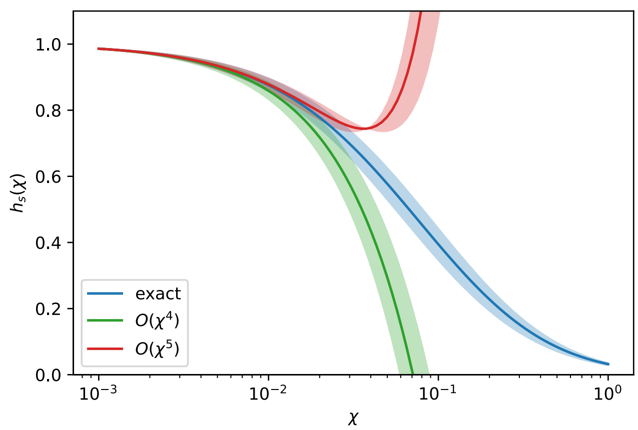


FIG. 2. Gaunt factor $\mathfrak{g}_s$ (upper) and analog quantity $\mathfrak{h}_s$ for the spreading (lower). The bands indicate the strength of the spin effect, $-1 \le s_B \le +1$. Curves have the following meaning: exact [blue, Eqs. (62) and (65)] and asymptotics for small χ up to χ^3 (orange), χ^4 (green), and χ^5 (red).

leading order ultrarelativistic approximation of the classical RR force in the Herrera form. This immediately indicates how to restore $\dot{u} \cdot u = 0$ for Eq. (59),

$$\dot{u}^\mu = qF^{\mu\nu}u_\nu + \tau_R \mathfrak{g}_s \Delta^\mu_\nu F^{\nu\lambda}F_{\lambda\kappa}u^\kappa . \tag{64}$$

Similarly, our Eq. (60) does not contain the classical RR corrections to spin-precession of Eq. (11). From Eq. (11), it is clear that only contribution to $\delta\Omega_{\rm RR}$ is from the component of RR perpendicular to $\boldsymbol{p}$. The collinear emission assumption, thus, precludes any corrections of this kind. However, as we discussed in Sec. II C, the effects of $\delta\Omega_{\rm RR}$ are negligible for typical high-intensity laser–plasma and laser–beam interactions.

It would be interesting to investigate whether one automatically gets the correct classical limit if the angular distribution of photon emission is taken into account correctly in the collision operators instead of relying on the collinear emission approximation.[67,86]

By analogy to $\mathfrak{g}_s$, we can also define a "Gaunt factor" $\mathfrak{h}_s$ for the spreading effect as the ratio of $K_0 + \boldsymbol{s}\cdot\boldsymbol{K}_i$ and its leading asymptotic term for $\chi \to 0$, $K_0 \sim \frac{\alpha m \chi^3 55}{24\sqrt{3}}$ as

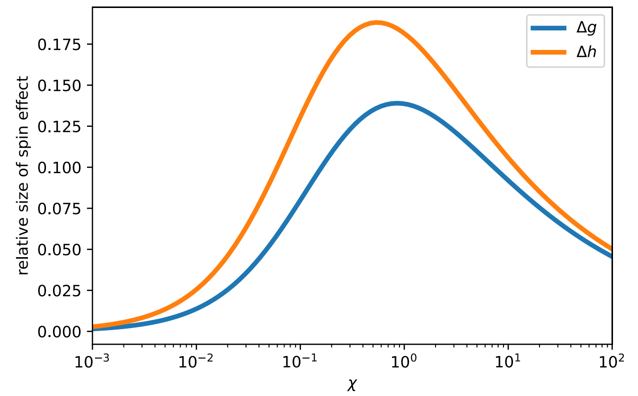


FIG. 3. Relative size of the spin effect on the Gaunt factors $\mathfrak{g}_s$ and $\mathfrak{h}_s$.

$$\begin{aligned}\mathfrak{h}_s(\chi) &= -\frac{24\sqrt{3}}{55\chi^3}\int_0^1 {\rm d}\lambda\lambda^2\left({\rm Ai}_1(z) + 2g\frac{{\rm Ai}'(z)}{z} + \lambda\frac{{\rm Ai}(z)}{\sqrt{z}}(\boldsymbol{s}\cdot\hat{\mathcal{B}})\right)\\ &\simeq 1 - \left(\frac{448\sqrt{3}}{55} + \frac{63}{22}s_B\right)\chi + \left(\frac{777}{4} + \frac{480\sqrt{3}}{11}s_B\right)\chi^2. \end{aligned}\tag{65}$$

The asymptotic expansion for small χ shows that the spreading is reduced if the electrons have positive s_B and increased if s_B is negative. The function $\mathfrak{h}_s$ and its asymptotic approximation for small χ are plotted in the lower panel of Fig. 2.

To better quantify the relative size of the spin effect on $\mathfrak{g}_s$ and $\mathfrak{h}_s$, we plot in Fig. 3 the quantities $\Delta\mathfrak{g} \equiv (\mathfrak{g}_{-1} - \mathfrak{g}_{+1})/(\mathfrak{g}_{-1} + \mathfrak{g}_{+1})$ and $\Delta\mathfrak{h} \equiv (\mathfrak{h}_{-1} - \mathfrak{h}_{+1})/(\mathfrak{h}_{-1} + \mathfrak{h}_{+1})$. Both quantities vanish in the classical limit $\chi \ll 1$ and peak at $\chi \sim 1$ with values of $\sim$0.14 and 0.19, respectively. For even larger χ, the relative spin contribution decreases.

B. Evolution of average energy and average energy spread, and energy-polarization correlation

To define an average of a quantity such as the average energy of a plasma particle, we need to divide the corresponding phase space integral by a suitable number density. This choice is not unique and several definitions exist in the literature. To facilitate comparison of our spin-dependent results with spin-independent ones from the literature,[32] we will choose here to divide by the zero-component of the current four-vector $n = J^0$, corresponding to the laboratory frame density. This choice means that the equations lose their Lorentz covariance, but are more convenient to work with than other choices.

We define a laboratory frame *average* of a quantity X as follows:

$$\bar{X} \equiv \frac{\langle \epsilon_p X\rangle}{n} = \frac{\langle \epsilon_p X\rangle}{\langle \epsilon_p\rangle} . \tag{66}$$

Thus, the average fluid velocity is $\bar{v}^\mu = J^\mu/n = (1, \bar{\boldsymbol{v}})$, where $\bar{\boldsymbol{v}} = \int {\rm d}^3\boldsymbol{p}\,\boldsymbol{v}f/\int {\rm d}^3\boldsymbol{p}f$ with $\boldsymbol{v} = \boldsymbol{p}/\epsilon_p$. We should note, however, that $\bar{v}^\mu$ is not a valid definition of a fluid four-velocity as it does not square to unity; $\bar{v}_\mu\bar{v}^\mu = 1 - \bar{\boldsymbol{v}}^2 < 1$. However, we can define a Lorentz factor, $\bar{\gamma} = (1 - \bar{\boldsymbol{v}}^2)^{-1/2}$ such that $u^\mu \equiv \bar{\gamma}\bar{v}^\mu$ agrees with the commonly used Eckart frame fluid four-velocity[70] and $n = \bar{\gamma}\sqrt{J\cdot J}$, see also Appendix C.

We can now define an average four-momentum as $\bar{p}^\mu = T^{0\mu}/n$, average energy $\bar{\epsilon}_p = T^{00}/n$ (equivalent to an "effective temperature" for the species), and average spin polarization as $\bar{\boldsymbol{s}} = \boldsymbol{J}_s^0/n = \langle \epsilon_p \boldsymbol{s}\rangle/\langle \epsilon_p\rangle$. Moreover, the average squared energy is $\overline{\epsilon_p^2} = U^{000}/n$.

With this, we first note that the current conservation yields the following auxiliary equation:

$$d_t n + n\nabla\cdot\bar{\boldsymbol{v}} = 0, \tag{67}$$

having introduced the convective derivative $d_t = \partial_t + \bar{\boldsymbol{v}}\cdot\nabla$.

From Eq. (44), we can derive equations for the average energy and momentum. We will focus here on the average energy, i.e., $\nu = 0$ component,

$$n d_t \bar{\epsilon}_p = -\nabla\cdot\boldsymbol{Q} + q\boldsymbol{J}\cdot\boldsymbol{E} - \langle \epsilon_p(I_0 + \boldsymbol{s}\cdot\boldsymbol{I}_i)\rangle, \tag{68}$$

with $\delta\epsilon = \epsilon_p - \bar{\epsilon}_p$ and $\boldsymbol{Q} = n\overline{\boldsymbol{v}\delta\epsilon} = \langle \boldsymbol{p}\delta\epsilon\rangle$ is the spatial part of the intrinsic heat flux four-vector Q^μ, which is related to the total heat flux four-vector $Q_T{}^\mu = \langle \epsilon_p p^\mu\rangle$ through $Q^\mu = Q_T{}^\mu - \bar{\epsilon}_p J^\mu$. The first term on the right-hand side represents the energy diffusion (i.e., divergence of the heat flux), the second term represents Ohmic energy gain, and the third term represents the radiative energy loss, including the spin dependence.

Next we want to derive an equation for the evolution of variance of the energy, i.e., the second central moment $\sigma_\epsilon^2 = \overline{\epsilon_p^2} - \bar{\epsilon}_p{}^2$. From Eq. (47), we first derive the equation for $\overline{\epsilon_p^2}$ as

$$\begin{aligned} n d_t\overline{\epsilon_p^2} &= -\nabla\cdot\langle \boldsymbol{p}(\epsilon_p^2 - \overline{\epsilon_p^2})\rangle + q\boldsymbol{Q}\cdot\boldsymbol{E} \\ &\quad + \langle \epsilon_p^2(K_0 + \boldsymbol{s}\cdot\boldsymbol{K}_i)\rangle - 2\langle \epsilon_p^2(I_0 + \boldsymbol{s}\cdot\boldsymbol{I}_i)\rangle. \end{aligned} \tag{69}$$

With $d_t\sigma_\epsilon^2 = d_t\overline{\epsilon_p^2} - 2\bar{\epsilon}_p d_t\bar{\epsilon}_p$ and employing Eq. (68), we find for the energy variance,

$$\begin{aligned} n d_t\sigma_\epsilon^2 &= -\nabla\cdot\langle \boldsymbol{p}(\delta\epsilon^2 - \sigma_\epsilon^2)\rangle + 2(q\boldsymbol{E} - \nabla\bar{\epsilon}_p)\cdot\boldsymbol{Q} \\ &\quad + \langle \epsilon_p^2(K_0 + \boldsymbol{s}\cdot\boldsymbol{K}_i)\rangle - 2\langle \epsilon_p\delta\epsilon(I_0 + \boldsymbol{s}\cdot\boldsymbol{I}_i)\rangle. \end{aligned} \tag{70}$$

The first term on the right-hand side represents some measure of the transport of energy spread. With the interpretation of the average energy $\bar{\epsilon}_p$ as an effective temperature of the electrons, the term $\nabla\bar{\epsilon}_p$ can be interpreted as thermoelectric effective force. Analogous to the electric field acting on the current leading to energy gain/loss in Eq. (68), the electric field and thermoelectric term act on the heat flux to increase the energy spread. The last two terms govern how the energy spreading of the plasma particles changes due to radiation effects including the particle polarization. According to Ref. 32, where the corresponding terms were discussed for unpolarized electrons, the second-to-last term is the radiative heating of the beam due to the stochasticity of photon emission. The last term is a radiative beam cooling due to a phase space contraction because radiation losses are larger for higher-energy particles.

For the evolution of average spin and the spin-energy moment, we find

$$\begin{aligned} n d_t\bar{\boldsymbol{s}} &= -\nabla\cdot\Sigma + q\langle \boldsymbol{s}\times\Omega\rangle \\ &\quad + \langle \boldsymbol{W}_f - \boldsymbol{W}_i + \boldsymbol{s}\cdot\underline{W}_{if} - \boldsymbol{s}W_0\rangle, \end{aligned} \tag{71}$$

$$\begin{aligned} n d_t\overline{\epsilon_p\boldsymbol{s}} &= -\nabla\cdot\langle \boldsymbol{p}(\epsilon_p\boldsymbol{s} - \overline{\epsilon_p\boldsymbol{s}})\rangle + q\boldsymbol{J}_s\cdot\boldsymbol{E} + q\langle \epsilon_p\boldsymbol{s}\times\Omega\rangle \\ &\quad + \langle \epsilon_p(\boldsymbol{W}_f - \boldsymbol{W}_i + \boldsymbol{s}\cdot\underline{W}_{if} - \boldsymbol{s}W_0)\rangle \\ &\quad - \langle \epsilon_p(\boldsymbol{I}_f + \boldsymbol{s}\cdot\underline{I}_{if})\rangle, \end{aligned} \tag{72}$$

where we have defined the spin diffusion tensor $\Sigma = n\overline{\boldsymbol{v}\delta\boldsymbol{s}} = \langle \boldsymbol{p}\delta\boldsymbol{s}\rangle$, with $\delta\boldsymbol{s} = \boldsymbol{s} - \bar{\boldsymbol{s}}$. This yields for the covariance between average spin-polarization and average energy, $\mathrm{cov}(\epsilon_p, \boldsymbol{s}) = \overline{\epsilon_p\boldsymbol{s}} - \bar{\epsilon}_p\bar{\boldsymbol{s}}$, the following evolution equation:

$$\begin{aligned} n\, d_t\mathrm{cov}(\epsilon_p, \boldsymbol{s}) &= -\nabla\cdot\langle \boldsymbol{p}(\delta\epsilon\delta\boldsymbol{s} - \mathrm{cov}(\epsilon_p, \boldsymbol{s}))\rangle \\ &\quad + (q\boldsymbol{E} - \nabla\bar{\epsilon}_p)\cdot\Sigma - \boldsymbol{Q}\cdot\nabla\bar{\boldsymbol{s}} \\ &\quad + q\langle \boldsymbol{s}\times\Omega\delta\epsilon\rangle + \langle (\boldsymbol{W}_f - \boldsymbol{W}_i + \boldsymbol{s}\cdot\underline{W}_{if} - \boldsymbol{s}W_0)\delta\epsilon\rangle \\ &\quad - \langle \epsilon_p(\boldsymbol{I}_f + \boldsymbol{s}\cdot\underline{I}_{if})\rangle + \bar{\boldsymbol{s}}\,\langle \epsilon_p(I_0 + \boldsymbol{s}\cdot\boldsymbol{I}_i)\rangle. \end{aligned}$$

In addition to a thermoelectric force driving the spin diffusion Σ, this equation also contains a force due to spin gradients $\nabla\bar{\boldsymbol{s}}$ coupling to the heat flux. The interpretation of the remaining terms is as follows: The terms in the third line stem from spin precession and radiative polarization, respectively, and affect the correlation between spin and particle energy in a significant way if the particle energy spread $\delta\epsilon$ is large. The two terms in the last line are coming from spin-dependent radiation reaction. If we denote the contributions from this last line by $n\delta_I\mathrm{cov}(\epsilon_p, \boldsymbol{s})$ and expand the spin-vector along the principal directions, $\boldsymbol{s} = s_B\hat{\mathcal{B}} + s_E\hat{\mathcal{E}} + s_k\hat{\mathcal{K}}$, we obtain

$$\delta_I\mathrm{cov}(\epsilon_p, s_B) = -\overline{I_f} - \overline{\delta s_B I_0} + \bar{s}_B\overline{s_B I_i} - \overline{s_B I_3}, \tag{73}$$

$$\delta_I\mathrm{cov}(\epsilon_p, s_E) = -\overline{\delta s_E I_0} + \bar{s}_E\overline{s_B I_i} - \overline{s_E I_3}, \tag{74}$$

$$\delta_I\mathrm{cov}(\epsilon_p, s_K) = -\overline{\delta s_K I_0} + \bar{s}_K\overline{s_B I_i} - \overline{s_K I_4}, \tag{75}$$

which demonstrates the coupling of the different spin-polarization components; s_B affects the correlations of s_E and s_K, but not the other way around. In particular, for an initially unpolarized distribution, only Eq. (73) is nonzero.

C. Sokolov–Ternov effect

To discuss the Sokolov–Ternov effect from our kinetic equations, let us assume we have a globally constant direction $\hat{\mathcal{B}}$ for all electrons. This could be, for instance, a uniform and homogeneous magnetic field $\boldsymbol{B} = B_0\hat{z}$, with electrons circulating in a steady state with constant energy on cyclotron orbits and with $p_z = 0$. Sokolov and Ternov[79] introduced the fractions n^σ of electrons with spin-up ($\sigma = +1$) and spin-down ($\sigma = -1$) and formulated rate equations for those quantities involving the spin-flip rates.

Here, we can then define the number density of electrons with spin up/down, n^σ, analogous to Eq. (27) as

$$n^\sigma = \frac{1}{2}\int \mathrm{d}^3\boldsymbol{p}(f + \sigma a_z) = \frac{J^0 + \sigma\hat{\mathcal{B}}\cdot\boldsymbol{J}_s^0}{2}. \tag{76}$$

Since the integrated collision integral for f vanishes, i.e., the current J^μ is conserved, as was shown above, the evolution equation for n^σ is essentially given by the momentum integral over $\mathcal{C}_a$ as in Eq. (49),

$$\left(\frac{\partial n^\sigma}{\partial t}\right)_{\mathrm{rad}} = \frac{\sigma}{2}\int \frac{\mathrm{d}^3\boldsymbol{p}}{\epsilon_p} f(\boldsymbol{p})[W_f - W_i + s_z W_3]. \tag{77}$$

This is not exactly identical to the textbook treatment of the Sokolov–Ternov effect, since on the right-hand side of the equation, it is not straightforward to identify the densities n^σ. Instead, the distributions f and polarization degree s_z are weighted with the photon emission rates. In a homogeneous magnetic field, we have $\chi = |\boldsymbol{p}|B_0$, and thus, different parts of the distribution function have different photon emission rates. That means, different parts of the distribution spin-polarize at different rate.

If we are looking for a stationary state of the polarization degree, then we should require that the integrand of on the right-hand side of Eq. (77) must vanish for each value of $\boldsymbol{p}$ separately. This then leads to the Sokolov–Ternov equilibrium for each fraction of particles with parameter χ. Using the asymptotic series of the rates for small χ in Appendix B, we find

$$s_z^{eq}(\chi) \sim \frac{W_i - W_f}{W_3} \simeq -\frac{8}{5\sqrt{3}} + \frac{13}{50}\chi - \frac{92\sqrt{3}}{125}\chi^2 \,. \tag{78}$$

The leading term is the textbook Sokolov–Ternov result.[79] The χ-dependent corrections predict that the equilibrium polarization degree decreases with increasing χ. This phenomenon has been observed numerically for the radiative polarization of electrons in a rotating electric field, as shown in Refs. 4 and 7. Of course, the exact dynamics of spin polarization is more involved due to the temporal change of χ due to radiation reaction.

VII. SUMMARY

In this paper, we have derived the kinetic equations for a spin-polarized relativistic plasma with the leading quantum effects due to the interaction of the leptons with a strong field taken into account via Boltzmann-type collision operators,

$$\mathcal{T}f = \mathcal{C}_f\,, \quad \mathcal{T}\boldsymbol{a} = q\boldsymbol{a}\times\Omega + \mathcal{C}_{\boldsymbol{a}}, \tag{79}$$

for the scalar density f and the spin (axial-vector) density $\boldsymbol{a}$. The transport operator $\mathcal{T}$ is given in Eq. (20), the collision operators by Eqs. (33) and (34), and the precession vector Ω by Eq. (10), with the field-dependent anomalous moment $a_e = a_e(\chi)$. The local quantum collision operators were derived using the locally constant field approximation of the photon emission rates and electron self-energy expressions. The loop contribution provides exactly the anomalous precession.

Starting from a classical description of radiation reaction, we discussed the relevance of radiation reaction for the covariant BMT equation and found a RR correction term to spin-precession. While this term is crucial for the evolution of the covariant spin-vector S^μ, under typical conditions of high-intensity laser–plasma interactions, this correction term is not relevant for the dynamics of the spin-vector $\boldsymbol{s}$ in the electron rest frame.

From the kinetic equations, we have derived effective single-particle equations of motion for a spinning particle, taking into account the spin-dependent radiation reaction and radiative polarization. We derived the moment hierarchy and obtained equations for the mean energy, energy spreading, and the energy-spin correlation.

In this paper, we focused on the interplay between the spin polarization and radiation reaction effects. Hence, the discussion was restricted to a polarized electron plasma without pair production effects taken into account. A generalization of our results to achieve a consistent description of a QED plasma of electrons, positrons, and photons should be straightforward.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Daniel Seipt: Conceptualization (lead); Formal analysis (lead); Methodology (equal); Writing – review & editing (equal). **Alexander G. R. Thomas:** Conceptualization (supporting); Formal analysis (supporting); Methodology (equal); Writing – review & editing (equal).

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

APPENDIX A: POLARIZATION RESOLVED PHOTON EMISSION RATES

Here, we summarize all the polarization resolved photon emission rates appearing in the collision operators. Following Ref. 58, we define the differential rate for the collinear emission of a photon with momentum $\boldsymbol{k}$ by an electron with momentum $\boldsymbol{p}$ as

$$w_A(\boldsymbol{p},\boldsymbol{k}) = \int_0^1 \mathrm{d}\lambda\, \omega_k \delta^{(3)}(\boldsymbol{k} - \lambda\boldsymbol{p})\frac{\mathrm{d}R_A}{\mathrm{d}\lambda}, \tag{A1}$$

where λ is the fractional momentum transfer from the electron to the photon, and the label A denotes the various polarization dependent contributions. The corresponding total rates are given by

$$W_A(\boldsymbol{p}) = \int \frac{\mathrm{d}^3\boldsymbol{k}}{\omega_k} w_A(\boldsymbol{p},\boldsymbol{k}) = \int_0^1 \mathrm{d}\lambda\,\frac{\mathrm{d}R_A}{\mathrm{d}\lambda} = R_A, \tag{A2}$$

where $\mathrm{d}^3\boldsymbol{k}/\omega_k$ is the Lorentz invariant phase space element of the emitted photon, with $\omega_k = |\boldsymbol{k}|$. The $\mathrm{d}R_A/\mathrm{d}\lambda$ are the angularly-integrated LCFA rate building blocks for the polarization dependent photon emission rate (see, e.g., Ref. 22) that are given here as a probability per unit proper time as

$$\frac{\mathrm{d}R_0}{\mathrm{d}\lambda} = -\alpha\left[\mathrm{Ai}_1(z) + 2g\frac{\mathrm{Ai}'(z)}{z}\right], \tag{A3}$$

$$\frac{\mathrm{d}\boldsymbol{R}_i}{\mathrm{d}\lambda} = -\alpha\lambda\frac{\mathrm{Ai}(z)}{\sqrt{z}}\hat{\mathcal{B}} = \frac{\mathrm{d}R_i}{\mathrm{d}\lambda}\hat{\mathcal{B}}, \tag{A4}$$

$$\frac{\mathrm{d}\boldsymbol{R}_f}{\mathrm{d}\lambda} = -\alpha\frac{\lambda}{1-\lambda}\frac{\mathrm{Ai}(z)}{\sqrt{z}}\hat{\mathcal{B}} = \frac{\mathrm{d}R_f}{\mathrm{d}\lambda}\hat{\mathcal{B}}, \tag{A5}$$

$$\begin{aligned}\frac{\mathrm{d}R_{if}}{\mathrm{d}\lambda} &= \frac{\mathrm{d}R_1}{\mathrm{d}\lambda}(\hat{\mathcal{B}}\hat{\mathcal{B}} + \hat{\mathcal{E}}\hat{\mathcal{E}}) + \frac{\mathrm{d}R_2}{\mathrm{d}\lambda}\hat{\mathcal{K}}\hat{\mathcal{K}} \\ &= \frac{\mathrm{d}R_0}{\mathrm{d}\lambda}\mathbf{1} + \frac{\mathrm{d}R_3}{\mathrm{d}\lambda}(\hat{\mathcal{B}}\hat{\mathcal{B}} + \hat{\mathcal{E}}\hat{\mathcal{E}}) + \frac{\mathrm{d}R_4}{\mathrm{d}\lambda}\hat{\mathcal{K}}\hat{\mathcal{K}}\,,\end{aligned} \tag{A6}$$

$$\frac{\mathrm{d}R_1}{\mathrm{d}\lambda} = -\alpha\left[\mathrm{Ai}_1(z) + 2\frac{\mathrm{Ai}'(z)}{z}\right], \tag{A7}$$

$$\frac{\mathrm{d}R_2}{\mathrm{d}\lambda} = -\alpha\left[(2g-1)\mathrm{Ai}_1(z) + 2g\frac{\mathrm{Ai}'(z)}{z}\right], \tag{A8}$$

$$\frac{\mathrm{d}R_3}{\mathrm{d}\lambda} = +\alpha\frac{\lambda^2}{1-\lambda}\frac{\mathrm{Ai}'(z)}{z}, \tag{A9}$$

$$\frac{\mathrm{d}R_4}{\mathrm{d}\lambda} = -\alpha\frac{\lambda^2}{1-\lambda}\mathrm{Ai}_1(z)\,. \tag{A10}$$

The label $A=0$ stands for the contribution of unpolarized particles, and $A=i(f)$ is for initially(finally) polarized electrons. The different polarization correlation contributions ($A=if$) are be denoted by $A=1,2,3,4$. Ai is the Airy function with argument $z=(\lambda/\chi(1-\lambda))^{2/3}$, Ai $'$ its derivative, and Ai_1 the integral $\mathrm{Ai}_1(z)=\int_0^\infty \mathrm{d}x\,\mathrm{Ai}(x+z)$, $g=1+\frac{\lambda^2}{2(1-\lambda)}$, and χ is the quantum parameter, which is expressed as follows:

$$\chi = \sqrt{p\cdot F^2\cdot p} = \sqrt{(\epsilon_p \boldsymbol{E}(t,\boldsymbol{x}) + \boldsymbol{p}\times\boldsymbol{B}(t,\boldsymbol{x}))^2 - (\boldsymbol{p}\cdot\boldsymbol{E}(t,\boldsymbol{x}))^2}. \tag{A11}$$

The principal directions $\hat{\mathcal{B}},\hat{\mathcal{E}}$ appearing in the above rates are unit vectors in the direction of the magnetic field and the electric field in the rest frame of the electron, and $\hat{\mathcal{K}}=\hat{\mathcal{E}}\times\hat{\mathcal{B}}$. In the ultra-relativistic approximation, $\epsilon_p\gg 1$, and when the angle between $\boldsymbol{p}$ and the field is not small, we can write

$$\hat{\mathcal{E}} = -q\frac{\boldsymbol{E}+\hat{\boldsymbol{p}}\times\boldsymbol{B}-\hat{\boldsymbol{p}}(\hat{\boldsymbol{p}}\cdot\boldsymbol{E})}{\sqrt{(\hat{\boldsymbol{p}}\times\boldsymbol{E})^2+(\hat{\boldsymbol{p}}\times\boldsymbol{B})^2-2\hat{\boldsymbol{p}}\cdot(\boldsymbol{E}\times\boldsymbol{B})}}, \tag{A12}$$

$$\hat{\mathcal{B}} = -q\frac{\boldsymbol{B}-\hat{\boldsymbol{p}}\times\boldsymbol{E}-\hat{\boldsymbol{p}}(\hat{\boldsymbol{p}}\cdot\boldsymbol{B})}{\sqrt{(\hat{\boldsymbol{p}}\times\boldsymbol{E})^2+(\hat{\boldsymbol{p}}\times\boldsymbol{B})^2-2\hat{\boldsymbol{p}}\cdot(\boldsymbol{E}\times\boldsymbol{B})}}, \tag{A13}$$

where $\hat{\boldsymbol{p}}=\boldsymbol{p}/|\boldsymbol{p}|$. For positrons, with $q=+1$, the principal directions point opposite to the fields in the particle rest frame. Under this approximation, in the rest frame of the particle, the electromagnetic field appears to be a crossed field; thus, $\hat{\mathcal{E}}\perp\hat{\mathcal{B}}$, and $\hat{\mathcal{E}}$, $\hat{\mathcal{B}}$, $\hat{\mathcal{K}}$ form an orthonormal basis. Moreover, because of the collinear emission approximation, these unit vectors are the same for incident and final particles.

The photon emission probability for polarized electrons can be compactly expressed employing the Müller matrix M and the Stokes vectors $\boldsymbol{n}_{i,f}$ of the incident and final electrons as[22,51]

$$\frac{\mathrm{d}R}{\mathrm{d}\lambda} = \frac{1}{2}N_i M N_f^T, \tag{A14}$$

where $N_{i,f}=(1,\boldsymbol{n}_{i,f})$. It should be noted that $N_{i,f}$ are not Minkowski four-vectors, despite having four components.

The Müller matrix itself is constructed by collecting the LCFA building blocks above into a 4×4 dimensional matrix emission probability rate,

$$M = \begin{pmatrix} \mathrm{d}R_0/\mathrm{d}\lambda & \mathrm{d}\boldsymbol{R}_f/\mathrm{d}\lambda \\ \mathrm{d}\boldsymbol{R}_i/\mathrm{d}\lambda & \mathrm{d}\underline{R}_{if}/\mathrm{d}\lambda \end{pmatrix}. \tag{A15}$$

Equivalent representations for the polarization resolved nonlinear Compton rates have been recently given in Ref. 21, and a subset of these rates for electrons spin-polarized along the B-field direction was also given in Ref. 8. The rates can be also represented in the form of a density matrix.[5]

In addition to the photon emission contribution at $O(\alpha)$, we also need the $O(\alpha)$ contribution from the one-loop electron self-energy. More precisely, these contributions are coming from the interference of the one-loop self-energy at $\mathcal{O}(\alpha)$ with the "free" propagation of the electron. The expression is given by[22]

$$R_0^L = -R_0, \tag{A16}$$

$$\boldsymbol{R}_i^L = \boldsymbol{R}_f^L = \alpha\int \mathrm{d}\lambda\,\lambda\frac{\mathrm{Ai}(z)}{\sqrt{z}}\hat{\mathcal{B}}, \tag{A17}$$

$$\underline{R}_{if}^L = R_0^L\mathbf{1}_3 + \alpha\int\mathrm{d}\lambda\,\lambda\frac{\mathrm{Gi}(z)}{\sqrt{z}}(\hat{\mathcal{K}}\hat{\mathcal{E}}-\hat{\mathcal{E}}\hat{\mathcal{K}}), \tag{A18}$$

where Gi is the Scorer function.[87] Airy and Scorer functions are the real and imaginary parts of the integral

$$\mathrm{Ai}(z)+i\mathrm{Gi}(z) = \frac{1}{\pi}\int_0^\infty \mathrm{d}t\,e^{i\left(zt+\frac{t^3}{3}\right)}. \tag{A19}$$

The Müller matrix M^L for the loop is constructed analogously to Eq. (A15) as[22]

$$M^L = \begin{pmatrix} R_0^L & \boldsymbol{R}_f^L \\ \boldsymbol{R}_i^L & \underline{R}_{if}^L \end{pmatrix}. \tag{A20}$$

APPENDIX B: ASYMPTOTICS OF THE PHOTON EMISSION SPECTRUM MOMENTS

The moments of the photon emission spectrum are defined as follows:

$$W_A = \int_0^1 \mathrm{d}\lambda\frac{\mathrm{d}R_A}{\mathrm{d}\lambda}, \tag{B1}$$

$$I_A = \int_0^1 \mathrm{d}\lambda\,\lambda\frac{\mathrm{d}R_A}{\mathrm{d}\lambda}, \tag{B2}$$

$$K_A = \int_0^1 \mathrm{d}\lambda\,\lambda^2\frac{\mathrm{d}R_A}{\mathrm{d}\lambda}, \tag{B3}$$

where the subscript A runs over the different building blocks of the spin-dependent photon emission rate. For small $\chi\to 0$, the asymptotic expansions of $Q\in\{W_A, I_A, K_A\}$ can be written as

$$Q = \alpha\sum_{n=0}^{\infty}\beta_n^{\{Q\}}\chi^n, \tag{B4}$$

where the coefficients $\beta_n^{\{Q\}}$ are presented in Table II.

APPENDIX C: A NOTE ON THE FLUID VELOCITY AND RELATIVISTIC HYDRODYNAMICS

Here, we briefly discuss how one could properly define a fluid velocity. According to the literature, e.g., Ref. 70, there are two commonly used options for defining the fluid velocity u^μ: (i) the Eckart frame via the vanishing of particle number/mass diffusion (i.e., via the particle number current) and (ii) the Landau frame via the vanishing of energy diffusion as the normalized eigenvector of $T^{\mu\nu}$.

TABLE II. Coefficients $\beta_n^{\{Q\}}$ of the asymptotic expansion as $\chi \to 0$ for all moments up to order χ^5.

	1	χ	χ^2	χ^3	χ^4	χ^5
n	0	1	2	3	4	5
W_0	0	$\frac{5}{2\sqrt{3}}$	$-\frac{4}{3}$	$\frac{35}{4\sqrt{3}}$	$-\frac{92}{3}$	$\frac{13475}{32\sqrt{3}}$
W_1	0	$\frac{5}{2\sqrt{3}}$	$-\frac{4}{3}$	$\frac{55}{8\sqrt{3}}$	$-\frac{56}{3}$	$\frac{6545}{32\sqrt{3}}$
W_2	0	$\frac{5}{2\sqrt{3}}$	$-\frac{4}{3}$	$\frac{175}{24\sqrt{3}}$	$-\frac{62}{3}$	$\frac{7469}{32\sqrt{3}}$
W_3	0	0	0	$-\frac{5\sqrt{3}}{8}$	12	$-\frac{1155\sqrt{3}}{16}$
W_4	0	0	0	$-\frac{35}{24\sqrt{3}}$	10	$-\frac{1001\sqrt{3}}{16}$
W_i	0	0	$-\frac{\sqrt{3}}{4}$	3	$\frac{-105\sqrt{3}}{8}$	200
W_f	0	0	$-\frac{\sqrt{3}}{4}$	2	$\frac{-105\sqrt{3}}{16}$	80
I_0	0	0	$\frac{2}{3}$	$-\frac{55}{8\sqrt{3}}$	32	$-\frac{8855}{16\sqrt{3}}$
I_1	0	0	$\frac{2}{3}$	$-\frac{55}{8\sqrt{3}}$	28	$-\frac{6545}{16\sqrt{3}}$
I_2	0	0	$\frac{2}{3}$	$-\frac{55}{8\sqrt{3}}$	$\frac{86}{3}$	$-\frac{6853}{16\sqrt{3}}$
I_3	0	0	0	0	-4	$\frac{385\sqrt{3}}{8}$
I_4	0	0	0	0	$-\frac{10}{3}$	$\frac{1001}{8\sqrt{3}}$
I_i	0	0	0	-1	$\frac{35\sqrt{3}}{4}$	-200
I_f	0	0	0	-1	$\frac{105\sqrt{3}}{16}$	-120
K_0	0	0	0	$\frac{55}{24\sqrt{3}}$	$-\frac{56}{3}$	$\frac{14245}{32\sqrt{3}}$
K_1	0	0	0	$\frac{55}{24\sqrt{3}}$	$-\frac{56}{3}$	$\frac{6545}{16\sqrt{3}}$
K_2	0	0	0	$\frac{55}{24\sqrt{3}}$	$-\frac{56}{3}$	$\frac{3311}{8\sqrt{3}}$
K_3	0	0	0	0	0	$-\frac{385\sqrt{3}}{32}$
K_4	0	0	0	0	0	$-\frac{1001}{32\sqrt{3}}$
K_i	0	0	0	0	$-\frac{35\sqrt{3}}{16}$	100
K_f	0	0	0	0	$-\frac{35\sqrt{3}}{16}$	80

The Eckart approach defines a fluid four-velocity through $J^\mu = N u^\mu$, with the fluid velocity u^μ ($u^2 = 1$), and N is the invariant number density, i.e., the number density measured in the fluid rest frame. Now, the particle momentum p^μ may be split into contributions parallel and perpendicular to u^μ as

$$p^\mu = \kappa_p u^\mu + \Delta^{\mu\nu} p_\nu, \tag{C1}$$

with $\Delta^{\mu\nu} \equiv \eta^{\mu\nu} - u^\mu u^\nu$ and $\kappa_p \equiv u_\mu p^\mu$. With this, we have $\langle p^\mu \rangle = \langle \kappa_p \rangle u^\mu + \langle p^\mu - \kappa_p u^\mu \rangle$, i.e., the vanishing of the second term means there is no particle diffusion current in the Eckart frame. The number density is defined as $N = \langle \kappa_p \rangle = J^\mu u_\mu = \sqrt{J^\mu J_\mu}$. By the primary definition, we can also use $u^\mu = J^\mu / \sqrt{J \cdot J}$ to define the fluid velocity. Particle number current conservation $\partial . J = 0$ yields the following equation: $\dot{N} + N\theta = 0$, where $\theta = \partial_\mu u^\mu$ is the expansion scalar (four-divergence of the fluid velocity), and $\dot{N} = (u^\mu \partial_\mu) N$ is the comoving or convective derivative.

The energy momentum tensor can be expanded into its irreducible tensor components as follows:

$$T^{\mu\nu} = \langle p^\mu p^\nu \rangle = e u^\mu u^\nu + u^\mu H^\nu + u^\nu H^\mu - p\Delta^{\mu\nu} + \pi^{\mu\nu}, \tag{C2}$$

with energy density $e = \langle \kappa_p^2 \rangle$, energy diffusion current (equivalent to heat flow in the Eckart description) $H^\mu = \langle \kappa_p \Delta^\mu_\nu p^\nu \rangle$, hydrostatic pressure $p = -\frac{1}{3}\Delta_{\mu\nu}\langle p^\mu p^\nu \rangle$, and shear-stress (or viscous pressure) tensor $\pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta}\langle p^\alpha p^\beta \rangle$, with the double-symmetric, traceless rank-4 projection operator orthogonal to u^μ,

$$\Delta^{\mu\nu}_{\alpha\beta} = \frac{1}{2}(\Delta^\mu_\alpha \Delta^\nu_\beta + \Delta^\mu_\beta \Delta^\nu_\alpha) - \frac{\Delta^{\mu\nu}\Delta_{\alpha\beta}}{\Delta^\lambda_\lambda}\,. \tag{C3}$$

Moreover, $\pi^\mu_\mu = 0$, $u_\mu \pi^{\mu\nu} = 0$ and $u_\mu H^\mu = 0$. Thus, the trace of the energy momentum tensor $T^\mu_\mu = e - 3p = \langle p^\mu p_\mu \rangle = \langle 1 \rangle = \int \frac{d^3 p}{\epsilon_p} f$.

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