Renormalization flow of nonlinear electrodynamics

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We study the renormalization flow of generic actions that depend on the invariants of the field strength tensor of an Abelian gauge field. While the Maxwell action defines a Gaussian fixed point, we search for further non-Gaussian fixed points or rather fixed functions, i.e., globally existing Lagrangians of the invariants. Using standard small-field expansion techniques for the resulting functional flow equation, a large number of fixed points is obtained, which—in analogy to recent findings for a shift-symmetric scalar field—we consider as approximation artifacts. For the construction of a globally existing fixed function, we pay attention to the use of proper initial conditions. Parametrizing the latter by the photon anomalous dimension, both the coefficients of the weak-field expansion are fully determined and those of the large-field expansion can be matched such that a global fixed function can be constructed for magnetic fields. The anomalous dimension also governs the strong-field limit. Our results provide evidence for the existence of a continuum of non-Gaussian fixed points parametrized by a small positive anomalous dimension below a critical value. We discuss the implications of this result within various scenarios with and without additional matter. For the strong-field limit of the 1PI QED effective action, where the anomalous dimension is determined by electronic fluctuations, our result suggests the existence of a singularity free strong-field limit, circumventing the standard conclusions connected to the perturbative Landau pole.

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I. INTRODUCTION

Relativistic models of nonlinear electrodynamics have an extensive history in field theory, beginning with Born- or Born-Infeld theory motivated by the removal of the divergence of the electron's self-energy in a classical setting [1–3], reemerging also as an effective theory of the open string [4]. The Heisenberg-Euler theory [5–8] represents not only the presumably correct theory of the nonlinear response of the electrodynamic quantum vacuum according to quantum electrodynamics (QED), but is also a hallmark of the concept of effective field theory now being ubiquitous in quantum field theory. Discovering the plethora of phenomena predicted by the Heisenberg-Euler action [9–12] is currently a substantial research endeavor in strong-field physics.

Perturbative renormalizability arguments suggest that nonlinear models of electrodynamics should not be viewed as a fundamental theory, as the nonlinear interactions are power-counting nonrenormalizable. Whether or not the naive perturbative conclusion can be extended to a strong coupling region is explored in the present work. In principle, perturbative arguments fail in presence of non-Gaussian fixed points which are a prerequisite for the construction of high-energy complete theories based on the concept of asymptotic safety [13,14].

Ultraviolet (UV) completeness of QED, i.e., including the fluctuations of fermionic or scalar charged particles, has been at the center of interest since the discovery of the perturbative Landau pole [15–17]. Though the Landau pole divergence of the coupling may or may not be an artifact of perturbation theory, there is consensus among various methods that the strong-coupling regime of conventional QED cannot be connected to the weak-coupling regime realized in nature because of chiral symmetry breaking and mass generation [18–22]. Beyond the conventional scenarios, UV completion in the large-flavor number limit [23–29], at a non-Gaussian Pauli coupling fixed point [30–33], novel resummation schemes [34], or UV completion mediated by gravitational fluctuations [35,36] have recently been discussed.

At first sight, it is thus not surprising that perturbative renormalization group (RG) resummations also find a

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Landau pole divergence in the strong-field limit of the higher-loop resummed Heisenberg-Euler effective action [37,37,38]. This holds both for the effective action as the 1PI generating functional, as well as for the Schwinger functional that include 1PR resummations [39–42]. At second glance, such a strong-field divergence may appear less plausible at least for the case of a homogeneous magnetic background. This is because the latter does not transfer energy to charged fluctuations and thus should not *per se* probe the high-momentum regime where Landau-pole singularities may play a role.

In the present work, we address the question of a possible existence of UV-complete nonlinear electrodynamics (without further charged matter degrees of freedom) as well as the strong-field limit of Heisenberg-Euler-type theories (with a minimum charged matter content), using methods of functional renormalization. More specifically, we derive the general nonperturbative RG flow equation for action functionals depending on the gauge- and Lorentz-invariant combinations of the field strength. For both aspects, we find that the criterion of *global existence* of functions satisfying the fixed-point equation for the action is most relevant. This is, in fact, familiar from technically similar searches for Wilson-Fisher-type fixed points [43–51], scaling solutions in fermionic/Yukawa theories [52–54], UV completions of gauged Higgs models [55,56] or gauged Yukawa models [57–59], and studies in quantum gravity [48,60–62].

As for the quest for matterless UV-complete asymptotically safe nonlinear electrodynamics, our answer is in the negative as long as standard search methods for fixed points based on *improper* initial conditions are used. For this case, our results are rather similar to analogous studies of shift-symmetric scalar theories or nonlinear Abelian models based on the Maxwell invariant [63,64]. While the weak-field expansion finds many potential fixed-point candidates similar to [63], the picture is quite comparable to the shift-symmetric scalar model, where the eigenperturbations around the fixed points are not integrable with respect to the induced measure [64]; therefore, only the trivial Gaussian fixed point remains in this analysis.

By contrast, we do find globally existing fixed-point actions, once the construction is based on *proper* initial conditions. We parametrize the latter in terms of the anomalous dimension of the photonic field which is either a free parameter, or could effectively be provided by charged matter fluctuations. With this parameter, we are able to construct a global action in the direction of one of the invariants by a nontrivial matching of the small- and large-field expansions. The approximations involved can be applied to the case of a purely magnetic field and thus provide evidence for the absence of Landau-pole-type singularities in the strong-field limit of this type.

The paper is structured as follows: in Sec. II, we introduce the setting for general theories of nonlinear electrodynamics including Minkowskian as well as Euclidean formulations. In Sec. III, we derive the RG flow on the considered theory space using the functional renormalization group. On a fixed point, the resulting flow equation reduces to a fixed-function equation, defining scaling solutions for generic effective Lagrangians. We also motivate and substantiate a set of approximations which simplify the analysis of the differential equation. Section IV is devoted to a standard analysis of the (reduced) fixed-function equation including the critical regime based on a conventional small-field expansion. Whereas this analysis corresponds to improper initial conditions, Sec. V investigates the fixed-function equation using proper initial conditions. A one-parameter family of global solutions is constructed on small- and large-field expansions for small positive anomalous dimensions. Our approximations are checked in Sec. VI by tackling the full partial differential equation in the small field regime. We interpret our results in the light of various scenarios in Sec. VII and conclude in Sec. VIII.

II. NONLINEAR ELECTRODYNAMICS

Maxwell's theory of electrodynamics (ED) in vacuum is a linear theory entailing a strict superposition principle. It can be defined in terms of the gauge potential $(\bar{\mathsf{A}}_\mu)$ in four-dimensional Minkowski space and the corresponding field strength tensor $(\bar{\mathsf{F}}_{\mu\nu})$ with its components being connected to the gauge potential in the usual way, $\bar{\mathsf{F}}_{\mu\nu} = \partial_\mu \bar{\mathsf{A}}_\nu - \partial_\nu \bar{\mathsf{A}}_\mu$.[1] Using the gauge and Lorentz invariant scalars formed from the field strength tensor and its Hodge dual $((\star\bar{\mathsf{F}})^{\mu\nu})$, that is

$$\bar{\mathcal{F}} \coloneqq \frac{1}{4}\bar{\mathsf{F}}_{\mu\nu}\bar{\mathsf{F}}^{\mu\nu}, \qquad \bar{\mathcal{G}} \coloneqq \frac{1}{4}\bar{\mathsf{F}}_{\mu\nu}(\star\bar{\mathsf{F}})^{\mu\nu} = \frac{1}{8}\varepsilon^{\mu\nu\rho\sigma}\bar{\mathsf{F}}_{\mu\nu}\bar{\mathsf{F}}_{\rho\sigma} \tag{1}$$

(using the convention $\varepsilon^{0123} = 1$), the free action reads

$$\mathsf{S}[\bar{\mathsf{A}}] = \int_{\mathbb{R}^{3,1}} -\bar{\mathcal{F}}(\bar{\mathsf{A}}(x))\mathrm{d}^4x. \tag{2}$$

Further local invariants involve derivatives of the field strength.

The most general effective action functional Γ of nonlinear electrodynamics may depend on all possible invariants; a generic theory can thus be parametrized by a local Lagrangian depending on the field strength and its derivatives:

[1] For the gauge field, the field strength, and the action, we consistently use a notation, where serifless fonts are used for Minkowski-valued quantities. The standard notation is reserved for the renormalized quantities on Euclidean space, which will be defined later. An overbar indicates an unrenormalized and typically dimensionful quantity.

$$\Gamma[\bar{\mathsf{A}}] = \int_{\mathbb{R}^{3,1}} \bar{\mathcal{L}}\big(\bar{\mathsf{F}}(x), \partial_\mu \bar{\mathsf{F}}(x), \partial_\nu \partial_\lambda \bar{\mathsf{F}}(x), \ldots\big) \mathrm{d}^4x. \tag{3}$$

In the present work, we ignore possible dependencies on the derivative terms and concentrate on the full functional dependence on the two invariants $\bar{\mathcal{F}}$ and $\bar{\mathcal{G}}$. This may be viewed as the leading order of a systematic derivative expansion of the action [65–70] in the spirit of the Heisenberg-Euler expansion [5]. However, in contradistinction to conventional derivative expansions where higher-order derivatives have to be small compared to a physical mass scale, our expansion is based on a comparison to a running RG scale k. The validity criterion therefore is that the derivative terms should have a small influence on the flow of the nonderivative terms at any scale k; they do not necessarily have to be numerically small. We thus approximate the general Lagrangian by an $\bar{\mathsf{F}}$-dependent function, or equivalently a function of the invariants, $\bar{\mathcal{L}}(\bar{\mathsf{F}}, \partial_\mu \bar{\mathsf{F}}, \ldots) \approx \bar{\mathcal{L}}(\bar{\mathsf{F}}, 0, \ldots) \equiv \bar{\mathcal{W}}(\bar{\mathcal{F}}(\bar{\mathsf{F}}), \bar{\mathcal{G}}(\bar{\mathsf{F}}))$, reducing the action to

$$\Gamma[\bar{\mathsf{A}}] \approx \int_{\mathbb{R}^{3,1}} \bar{\mathcal{W}}\big(\bar{\mathcal{F}}(\bar{\mathsf{A}}(x)), \bar{\mathcal{G}}(\bar{\mathsf{A}}(x))\big) \mathrm{d}^4x. \tag{4}$$

This defines the class of action functionals covering nonlinear generalizations of vacuum electrodynamics to leading-derivative order. The corresponding equations of motion generically represent hyperbolic second-order nonlinear partial differential equations for which initial-value problems can be formulated.

In the following, we restrict ourselves to parity-invariant theories. Since $\bar{\mathcal{G}}$ is parity-odd, $\bar{\mathcal{W}}$ should be considered as an even function of $\bar{\mathcal{G}}$, i.e., instead of $\bar{\mathcal{W}}(\bar{\mathcal{F}}, \bar{\mathcal{G}})$ we write $\bar{\mathcal{W}}(\bar{\mathcal{F}}, \bar{\mathcal{G}}^2)$.

As our renormalization group analysis will be performed in Euclidean spacetime, let us detail the connection between Minkowski-valued and Euclidean quantities: In $d = 4$ dimensional Minkowski space, the components of the antisymmetric field strength tensor, $\bar{\mathsf{F}}_{\mu\nu}$, are related to the electric and magnetic field components by $\bar{\mathsf{F}}_{0i} = \bar{\mathsf{E}}_i$ and $\bar{\mathsf{F}}_{ij} = \varepsilon_{ijl}\bar{\mathsf{B}}_l$. In terms of the fields, the invariant scalars read $\bar{\mathcal{F}} = \frac{1}{2}(\bar{\mathsf{B}}^2 - \bar{\mathsf{E}}^2)$ and $\bar{\mathcal{G}} = -\bar{\mathsf{E}} \cdot \bar{\mathsf{B}}$.

In the Euclidean, we start from a Euclidean gauge potential $(\bar{A}_\mu)$ with field strength components $\bar{F}_{\mu\nu} = \partial_\mu \bar{A}_\nu - \partial_\nu \bar{A}_\mu$ and the components of its Hodge dual $(\star \bar{F})^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\bar{F}_{\rho\sigma}$. The corresponding invariants read $\bar{\mathscr{F}} := \frac{1}{4}\bar{F}_{\mu\nu}\bar{F}^{\mu\nu}$ and $\bar{\mathscr{G}} := \frac{1}{4}\bar{F}_{\mu\nu}(\star\bar{F})^{\mu\nu}$, where the Euclidean metric is used for the contractions. Identifying the Euclidean field strength components as $\bar{F}_{0i} = \bar{E}_i$ and $\bar{F}_{ij} = \varepsilon_{ijl}\bar{B}_l$, we obtain for the invariants $\bar{\mathscr{F}} = \frac{1}{2}(\bar{B}^2 + \bar{E}^2)$ and $\bar{\mathscr{G}} = \bar{E} \cdot \bar{B}$.

On the level of the invariants $\bar{\mathcal{F}}$ and $\bar{\mathscr{F}}$, the transition between Minkowskian and Euclidean spacetime is captured by the field replacement rule $(\bar{\mathsf{E}}, \bar{\mathsf{B}}) \leftrightarrow (-\imath \bar{E}, \bar{B})$. This also implies a relation between $\bar{\mathcal{G}}$ and $\bar{\mathscr{G}}$, which in total yields: $\bar{\mathcal{F}} \leftrightarrow \bar{\mathscr{F}}$ and $\bar{\mathcal{G}} \leftrightarrow \imath\bar{\mathscr{G}}$.

Including the Wick rotation in coordinate space, the corresponding Euclidean action, e.g., of Maxwell's theory reads

$$S[\bar{A}] = \int_{\mathbb{R}^4} \bar{\mathscr{F}}\big(\bar{A}(x)\big) \mathrm{d}^4x. \tag{5}$$

Analogously to Eq. (4), the corresponding class of general nonlinear theories of electrodynamics investigated in this work is described by a Euclidean action

$$\Gamma[\bar{A}] := \int_{\mathbb{R}^4} \bar{\mathscr{W}}\Big(\bar{\mathscr{F}}\big(\bar{A}(x)\big), \bar{\mathscr{G}}\big(\bar{A}(x)\big)^2\Big) \mathrm{d}^4x, \tag{6}$$

where the function $\bar{\mathscr{W}}$ is the Euclidean analog of $\bar{\mathcal{W}}$. On the level of the Lagrangian, the transition from Euclidean back to Minkowskian spacetime is thus performed by the replacements $\bar{\mathscr{W}} \to -\bar{\mathcal{W}}$, $\bar{\mathscr{F}} \to \bar{\mathcal{F}}$, and $\bar{\mathscr{G}} \to -\imath\bar{\mathcal{G}}$. The latter implies $\bar{\mathscr{G}}^2 \to -\bar{\mathcal{G}}^2$.

III. RG FLOW AND FIXED FUNCTIONS

Let us list the ingredients for our functional RG analysis for theories based on the action in Eq. (6). For conceptual and technical details, we refer the reader to reviews on the functional RG [71–77].

A. Scale-dependent effective action

Using the functional RG, we quantize nonlinear electrodynamics in a Wilsonian sense momentum shell by momentum shell. Quantization over a finite amount of scales is always possible in the spirit of an effective field theory. In addition, we intend to search for fixed points, or rather fixed functions, of the RG flow that have the potential to allow for a consistent quantization on all scales. For this, we use the Wetterich equation [78–81] for a scale-dependent one-particle irreducible (1PI) effective action Γ_k,

$$k\partial_k \Gamma_k[\bar{A}] = \frac{1}{2}\mathbf{Tr}\Big[(\Gamma_k^{(2)} + \mathcal{R}_k)^{-1} k\partial_k \mathcal{R}_k\Big][\bar{A}], \tag{7}$$

where $\Gamma_k^{(2)}$ denotes the second functional derivative of Γ_k with respect to the gauge field $\bar{A}$. The quantity $\mathcal{R}_k$ is a regulator that controls infrared (IR) mode suppression below a momentum scale k and implements the Wilsonian momentum-shell integration. For a given initial condition, e.g., at a UV scale $\Gamma_{k=\Lambda}$, the action $\Gamma_{k=0}$ includes all quantum fluctuations with momenta below Λ [71–77].

In the present work, we parametrize the action functional Γ_k by a scale-dependent variant of the nonlinear theory space spanned by Eq. (6) amended by a Lorenz gauge-fixing term,

$$\Gamma_k[\bar{A}] \coloneqq \int_{\mathbb{R}^4}\left(\bar{\mathscr{W}}_k(\bar{\mathscr{F}},\bar{\mathscr{G}}^2)+\frac{1}{2\alpha}Z_k(\partial_\mu\bar{A}^\mu)^2\right)\mathrm{d}^4x, \qquad (8)$$

where we have suppressed the x dependencies.

We assume that the function $\bar{\mathscr{W}}_k$ features a weak-field expansion of the form $\bar{\mathscr{W}}_k(\bar{\mathscr{F}},\bar{\mathscr{G}}^2)=Z_k\bar{\mathscr{F}}+\cdots$, where Z_k can be interpreted as a wave function renormalization. We have included Z_k also in the gauge-fixing term in order to obtain a standard form of the gauge-fixed propagator including the gauge-fixing parameter $\alpha\in\mathbb{R}$. Analogous parametrizations of the effective action also including the non-Abelian case have been studied in [22,82–85].

The scale dependence of Z_k is encoded in the anomalous dimension of the gauge field,

$$\eta_k \coloneqq -k\partial_k \ln(Z_k). \qquad (9)$$

For the analysis of the RG flow, it is useful to introduce dimensionless renormalized quantities. In $d=4$ dimensions, the corresponding rescalings using the scale k read:

$$\mathscr{F}\coloneqq Z_k k^{-4}\bar{\mathscr{F}}, \qquad \mathscr{G}\coloneqq Z_k k^{-4}\bar{\mathscr{G}}, \qquad w_k\coloneqq k^{-4}\bar{\mathscr{W}}_k. \qquad (10)$$

The Z_k rescaling of the field implies that the weak-field expansion of the *field-strength potential* w_k starts with $w_k=\mathscr{F}+\cdots$. In accordance with Eq. (10), we also introduce a dimensionless-renormalized field strength and a conveniently rescaled (though dimensionful) gauge field:

$$F\coloneqq\sqrt{Z_k}k^{-2}\bar{F}, \qquad A\coloneqq\sqrt{Z_k}k^{-2}\bar{A}. \qquad (11)$$

Note that A carries an inverse mass dimension, such that the dimensionless field strength components $F_{\mu\nu}$ maintain their standard form, $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$, and the scale-dependent effective action yields

$$\Gamma_k[A]=k^4\int_{\mathbb{R}^4}\left(w_k(\mathscr{F},\mathscr{G}^2)+\frac{1}{2\alpha}(\partial_\mu A^\mu)^2\right)\mathrm{d}^4x. \qquad (12)$$

Here, we have once again suppressed the x dependencies under the integral.

B. RG flow equation

For the evaluation of the Wetterich equation, we need the Hessian of the action:

$$(\Gamma_k^{(2)})^{\mu\nu}[\bar{A}(A)](x,x')=Z_k k^{-4}\frac{\delta^2\Gamma_k[A]}{\delta A_\mu(x)\delta A_\nu(x')}. \qquad (13)$$

With respect to the continuous part of the spectrum, the Hessian can be diagonalized in momentum space, since it suffices to consider a homogeneous field strength in order to extract information about the flow of w_k. Then, using Eq. (12), the following decomposition of $\Gamma_k^{(2)}$ in terms of projectors, i.e., idempotent endomorphisms, acting on the Lorentz components in field space is useful:

$$\Gamma_k^{(2)}=Z_k k^2\left(X_k^T\mathbf{P}_T+X_k^L\mathbf{P}_L+\sum_{a=1}^{4}X_k^a\mathbf{P}_a\right). \qquad (14)$$

The projection operators and their coefficients are listed in Table I, using the shorthand notation

$$w_k'\coloneqq\frac{\partial w_k}{\partial\mathscr{F}} \quad \text{and} \quad \dot{w}_k\coloneqq\frac{\partial w_k}{\partial(\mathscr{G}^2)}. \qquad (15)$$

TABLE I. Algebraic expressions for projections and their respective coefficients according to the expansion in Eq. (14). The symbol $\otimes$ denotes the dyadic product on $\mathbb{R}^4\times\mathbb{R}^4$. Here, p is a dimensionful momentum space coordinate, and $y=p/k$ its dimensionless complement.

Projector	Coefficient
$\mathbf{P}_T=\mathbb{1}-\frac{p\otimes p}{p^2}$	$X_k^T=w_k'y^2$
$\mathbf{P}_L=\frac{p\otimes p}{p^2}$	$X_k^L=\frac{1}{\alpha}y^2$
$\mathbf{P}_1=\frac{(Fp)\otimes(Fp)}{(Fp)^2}$	$X_k^1=w_k''(Fy)^2$
$\mathbf{P}_2=\frac{(\star Fp)\otimes(\star Fp)}{(\star Fp)^2}$	$X_k^2=2(\dot{w}_k+2\mathscr{G}^2\ddot{w}_k)(\star Fy)^2$
$\mathbf{P}_3=\frac{(Fp)\otimes(\star Fp)}{\mathscr{G}p^2}$	$X_k^3=2\mathscr{G}^2\dot{w}_k'y^2$
$\mathbf{P}_4=\frac{(\star Fp)\otimes(Fp)}{\mathscr{G}p^2}$	$X_k^4=X_k^3$

For convenience, we have also introduced a dimensionless momentum space coordinate $y\coloneqq p/k$.

Let us elucidate some properties of the field space projection endomorphisms. The projectors act as linear operators on the space of gauge fields, that is the space of 1-form fields $\mathfrak{X}^*(\mathbb{R}^4)$ which contains smooth sections of the cotangent bundle $T^*\mathbb{R}^4$ with respect to the standard smooth structure on four-dimensional Euclidean space. The images of the projections are linear subspaces of $\mathfrak{X}^*(\mathbb{R}^4)$. In particular, $\mathbf{P}_T(\mathfrak{X}^*(\mathbb{R}^4))$ and $\mathbf{P}_L(\mathfrak{X}^*(\mathbb{R}^4))$ define a transversal and longitudinal component of $\mathfrak{X}^*(\mathbb{R}^4)$, respectively. They are orthogonal complements to each other, i.e., $\mathbf{P}_T\circ\mathbf{P}_L=\mathbf{P}_L\circ\mathbf{P}_T=0$. In fact, the corresponding involution $i_T\coloneqq\mathbb{1}-2\mathbf{P}_T$ induces a natural $\mathbb{Z}_2$-gradation for $\mathfrak{X}^*(\mathbb{R}^4)$. Since $\mathbb{1}=\mathbf{P}_T+\mathbf{P}_L$, an arbitrary element $A\in\mathfrak{X}^*(\mathbb{R}^4)$ can thus be decomposed into transversal and longitudinal parts:

$$A=\mathbb{1}(A)=\mathbf{P}_T(A)+\mathbf{P}_L(A)\equiv A_T+A_L. \qquad (16)$$

The class of projectors $\mathbf{P}_a$ for $a\in\{1,2,3,4\}$ refer to further subtransversal projections, because $\mathbf{P}_a\circ\mathbf{P}_L=\mathbf{P}_L\circ\mathbf{P}_a=0$. Due to $\mathbb{1}=\mathbf{P}_T+\mathbf{P}_L$, we have $\mathbf{P}_a(\mathfrak{X}^*(\mathbb{R}^4))\subseteq\mathbf{P}_T(\mathfrak{X}^*(\mathbb{R}^4))$ for all a. From compositions among subtransversal projections $\mathbf{P}_a$, it is further noted that they agree on an equal rank and likewise share the *same* one-dimensional image. Hence it is not possible to span the three-dimensional transversal subspace using a combination of the $\mathbf{P}_a$; however, we can still make use of their properties evaluating the explicit form of the flow and allowing us to estimate the weight of individual contributions to it.

Correspondingly, we span the regulator $\mathcal{R}_k$ with the aid of the transversal and longitudinal projectors, yielding in momentum space:

$$\mathcal{R}_k(y) = Z_k k^2 y^2 r(y^2)\left[\mathbf{P}_T + \frac{1}{\alpha}\mathbf{P}_L\right], \tag{17}$$

where the information about the details of the momentum-mode regularization are encoded in the dimensionless shape function r. Equations (14) and (17) read together form the inverse of the regularized propagator, $\Gamma_k^{(2)} + \mathcal{R}_k \equiv G_k^{-1}$, for which we now need the operator inverse. For this, we take advantage of the algebraic structure provided by the field space projections and expand the regularized full propagator as in Eq. (14):

$$\begin{aligned} G_k &= \left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1} \\ &= \frac{1}{Z_k k^2}\left(Y_k^T\mathbf{P}_T + Y_k^L\mathbf{P}_L + \sum_{a=1}^{4} Y_k^a\mathbf{P}_a\right). \end{aligned} \tag{18}$$

The coefficients Y_k^T, Y_k^L, Y_k^a for $a\in\{1,2,3,4\}$ are completely determined by the system of equations that follows from $\mathbb{1} = G_k^{-1}G_k$, using the composition table for the projections. The solution is

$$\begin{aligned} Y_k^T &= \frac{1}{X_k^T + y^2 r}, \qquad Y_k^L = \frac{1}{X_k^L + \frac{1}{\alpha}y^2 r}, \\ Y_k^1 &= Y_k^T\cdot\frac{(X_k^3)^2\xi^{-2} - X_k^1(X_k^T + X_k^2 + y^2 r)}{(X_k^T + X_k^1 + X_k^3 + y^2 r)(X_k^T + X_k^2 + X_k^3 + y^2 r) - (X_k^3 + X_k^2\xi^2)(X_k^1 + X_k^3\xi^{-2})}, \\ Y_k^2 &= Y_k^T\cdot\frac{(X_k^3)^2\xi^{-2} - X_k^2(X_k^T + X_k^1 + y^2 r)}{(X_k^T + X_k^1 + X_k^3 + y^2 r)(X_k^T + X_k^2 + X_k^3 + y^2 r) - (X_k^3 + X_k^2\xi^2)(X_k^1 + X_k^3\xi^{-2})}, \\ Y_k^3 &= Y_k^T\cdot\frac{X_k^1 X_k^2\xi^2 - X_k^3(X_k^T + X_k^3 + y^2 r)}{(X_k^T + X_k^1 + X_k^3 + y^2 r)(X_k^T + X_k^2 + X_k^3 + y^2 r) - (X_k^3 + X_k^2\xi^2)(X_k^1 + X_k^3\xi^{-2})}, \\ Y_k^4 &= Y_k^3, \end{aligned} \tag{19}$$

$$\text{with}\quad \xi(F,y) \coloneqq \frac{\mathscr{G}y^2}{\sqrt{(Fy)^2(\star Fy)^2}}.$$

As a useful consequence of the projection technique, the RHS of the flow equation (7) decomposes into a sum of traces over field space projectors.

Finally, the **Tr** operation in the flow equation runs over Lorentz indices and momentum space. Furthermore, because of field-strength homogeneity, the RG flow is projected onto the field-strength potential w_k. Introducing an RG time $t \coloneqq \ln(k/\Lambda)$, with an arbitrary reference scale $\Lambda\in\mathbb{R}^+$, the RG flow finally is described by an autonomous differential equation that reads:

$$\begin{aligned} &\partial_t w_k + 4w_k - (4+\eta_k)(w_k'\mathscr{F} + 2\dot{w}_k\mathscr{G}^2) \\ &\quad = -\frac{1}{32\pi^4}\int_{\mathbb{R}^4} y^2\left(\eta_k r(y^2) + 2y^2 r'(y^2)\right)Y_k(y)\mathrm{d}^4y, \end{aligned} \tag{20}$$

where r' denotes the derivative of r with respect to its argument y^2 and

$$Y_k \coloneqq 3Y_k^T + \frac{1}{\alpha}Y_k^L + Y_k^1 + Y_k^2 + 2Y_k^3. \tag{21}$$

Equation (20) generalizes previously derived flow equations for actions depending solely on $\mathscr{F}$ [22,86] to the general case of nonlinear ED, representing an important intermediate result of this work.

C. Fixed-point sector

In perturbative QED, the flow analogous to Eq. (20) develops singularities toward high energies, e.g., in the form of the Landau pole. By contrast, the RG flow can be UV complete if all operators spanning the action remain bounded. The latter can be realized with the aid of RG fixed points where the dimensionless flow vanishes and the theory develops a quantum scale symmetry [87]. The existence of such fixed points is a prerequisite for the asymptotic-safety scenario of quantum field theories.

In the following, we address the question as to whether the quantized version of nonlinear ED as described by the RG flow of Eq. (20) exhibits such a fixed point. If so, the scale derivative of the field-strength potential vanishes, $\partial_t w_k = 0$, and the potential approaches a *fixed function* $w_* \coloneqq w_{k\to\infty}$; in the language of statistical mechanics, w_* corresponds to a scaling function. A special focus on properties like nontriviality and global existence for the fixed function, if it exists, will be adopted later.

At the fixed point, w_* satisfies the fixed-function equation (FFE):

$$w_* - \left(1+\frac{\eta_*}{4}\right)(w_*'\mathscr{F} + 2\dot{w}_*\mathscr{G}^2) = -\frac{1}{128\pi^4}\int_{\mathbb{R}^4} y^2\big(\eta_* r(y^2) + 2y^2 r'(y^2)\big) Y_*(y)\mathrm{d}^4y. \tag{22}$$

Here and in the following, quantities evaluated at the fixed point are denoted with an asterisk. In particular, the quantity Y_* is obtained by evaluating Eq. (21) for $w_k = w_*$; note that Y_* contains derivatives of w_* up to second order in both arguments, such that Eq. (22) corresponds to a partial differential equation for w_* as a function of $\mathscr{F}$ and $\mathscr{G}^2$.

D. Approximations

In order to investigate the fixed-function equation (22), we use two approximations for simplifying the technical complexity:

(A1) The RHS of the FFE involves a momentum-space integral where spherical symmetry is broken by the directions of electric and magnetic fields. For instance, the term $(Fy)^2$ can be written as

$$(Fy)^2 = y_0^2E^2 + \vec{y}^2E^2\cos(\vartheta_E)^2 + \vec{y}^2B^2\sin(\vartheta_B)^2 + 2y_0\vec{y}\cdot(E\times B), \tag{23}$$

where we have used a Euclidean space-time decomposition of $y = (y_0, \vec{y})^\intercal$ with $\vec{y}\in\mathbb{R}^3$, and ϑ_E, ϑ_B denote the angles enclosed by $\vec{y}$ and E, B respectively. The angle dependence can be eliminated by (i) assuming that $\vartheta_E - \vartheta_B = n\pi$ for $n\in\mathbb{Z}$, implying that E and B are either parallel or antiparallel, and (ii) requiring $E^2 = B^2$. It can be shown, that these conditions are equivalent to *(anti-)self-dual nonlinear electrodynamics* for which $F = \pm(\star F)$ (where the minus sign corresponds to anti-self-duality); note that self-duality as used here is meaningful only in $d = 4$ dimensions. Using self-duality, Eq. (23) exhibits spherical symmetry in momentum space,

$$(Fy)^2 = \mathscr{F}y^2 = (\star Fy)^2, \tag{24}$$

such that the momentum integral in Eq. (22) can be done analytically for a variety of frequently used shape functions.

We emphasize that the choice of a self-dual field configuration for the evaluation of the RHS of the FFE does not yet represent an approximation. The RHS still contains the complete set of terms. With this choice, we, however, lose the ability to distinguish between dependencies of the RHS on the two different variables $\mathscr{F}$ or $\mathscr{G}$. In fact, since $\mathscr{G}^2 = \mathscr{F}^2$, every term of even power in $\mathscr{F}^2$ receives also contributions from the $\mathscr{G}^2$ dependence. Our first approximation (A1) therefore consists in accepting the (mis)identification of these terms on the RHS of the flow equation. As an advantage, the FFE now reduces to an ordinary differential equation as the bi-argument dependence of the fixed function w_* merges to a single-argument dependence solely on the invariant $\mathscr{F}$.

(A2) The FFE can also be transformed into an ordinary differential equation by truncating the theory space down to a pure $\mathscr{F}$ dependence of w_*, i.e., discarding the $\mathscr{G}^2$ dependence altogether. Then, the fact that $\dot{w}_* = \ddot{w}_* = \dot{w}_*' = 0$ implies that $Y_k^2 = Y_k^3 = Y_k^4 = 0$ according to Table I and using (19).

As a consequence, the scale-dependent propagator G_k receives only a single subtransversal contribution, cf. Eq. (18). Nevertheless, we believe that this truncation still provides a good approximation, since the subtransversal input arises from a unique one-dimensional subspace of $\mathfrak{X}^*(\mathbb{R}^4)$ and is likely to be of minor relevance compared to the full three-dimensional transversal input generated by $\mathbf{P}_T$. The latter is mediated through the coefficient Y_k^T which remains unaffected from this truncation. At the same time, this approximation yields a considerable simplification of Eq. (22).

Conversely, it would not be reasonable to restrict to a purely $\mathscr{G}^2$-dependent theory space and discard the $\mathscr{F}$ sector instead. This would eliminate the transversal input, only retaining one-dimensional contributions that would not cover the underlying four-dimensional field space. Also the Maxwell term would be discarded from the weak-field expansion of w_*, thereby losing a relevant part of theory space including the free theory and propagator.

We emphasize, that the approximations (A1) and (A2) are not equivalent. Clearly (A2) $\not\Rightarrow$ (A1), because a truncation of theory space does not induce any specification of the field strength used to build the invariants. (A1) is a restriction rather on the information extracted for the fixed function w_* than on the theory space. In addition, also (A1) $\not\Rightarrow$ (A2), since self-duality retains information about the $\mathscr{G}^2$ dependence by means of a projection on the $\mathscr{F}$-related subspace of theory space. In this manner, derivatives of w_* with respect to its $\mathscr{G}^2$ argument transform to $\mathscr{F}$ derivatives and, in particular, do not imply $\dot{w}_* = 0$. This is different from the demands of (A2).

In previous studies [22,86], the truncation represented by approximation (A2) has been applied. Specifically in [63], the problem of the angle dependence has been solved by an expansion and resummation technique. As we apply both (A1) and (A2), our resulting FFE differs from that of [63] by the terms kept from the identification of $\mathscr{G}^2 = \mathscr{F}^2$. On the other hand, by performing the momentum integration without expansion we have an unaffected access to the large-field limit of the FFE.

IV. FIXED FUNCTIONS FOR IMPROPER INITIAL CONDITIONS

In this section, we focus on the reduced FFE that we obtain based on the approximations (A1) and (A2). We search for solutions employing a weak-field expansion technique that is widely used in the literature, e.g., for the analysis of Wilson-Fisher-type fixed points in scalar field theories in a local-potential approximation [44,50,71,74,88–90], generic fermionic or Yukawa models [76,91–95], supersymmetric models [96–99], or asymptotically safe fixed points in gravity [100–106].

Even though this method is simple and has proven to lead to robust results in many examples, it is based on a choice of *improper* initial conditions for the FFE that do not fix the solution uniquely without additional assumptions and may, in fact, produce artifacts as will be discussed below. Indeed, our results are similar to those of [63,64] as we find many fixed-point candidates in addition to the Gaussian fixed point. Subsequently, we will, however, argue that *proper* initial conditions give a more immediate access to fixed-point candidates for the present system.

A. Reduced fixed-function equation

Applying both approximations (A2) and (A1) in this order reduces Eq. (22) significantly:

$$w_* - \left(1+\frac{\eta_*}{4}\right) w'_* \mathscr{F} = \frac{1}{64\pi^2}\left[3\mathrm{t}^4_{(10)}\left(w'_*;\frac{\eta_*}{2}\right) + \mathrm{t}^4_{(10)}\left(1;\frac{\eta_*}{2}\right) - w''_*\mathscr{F}\mathrm{t}^4_{(11)}\left(w'_*,(w'_*\mathscr{F})';\frac{\eta_*}{2}\right)\right], \quad (25)$$

where we have introduced the *threshold functions*

$$\mathrm{t}^d_{(n_1 n_2)}(z_1,z_2;a) \coloneqq \int_0^\infty y^{\frac{d}{2}-1}\frac{-yr'(y)-ar(y)}{\left(z_1+r(y)\right)^{n_1}\left(z_2+r(y)\right)^{n_2}}\mathrm{d}y. \quad (26)$$

If either $n_1 = 0$ or $n_2 = 0$, t does not depend on z_1 or z_2 respectively and we will just omit the redundant argument in our notation, e.g., $\mathrm{t}^d_{(n_1 0)}(z_1;a)$.

In the remainder of this section, we attempt to construct a solution to Eq. (25) using analytical techniques based on small-field expansions.

B. Small-field expansion

Assuming that the field-strength potential at the fixed point can be expanded in a Taylor series near the origin,

$$w_*(\mathscr{F}) = \sum_{i=0}^{\infty} u_{i,*}\mathscr{F}^i = u_{0,*} + \mathscr{F} + u_{2,*}\mathscr{F}^2 + O(\mathscr{F}^3) \quad \text{as } \mathscr{F}\to 0, \quad (27)$$

we need to determine the generalized fixed-point couplings $u_{i,*}$ from the FFE (25).

We implement the IR regularization using the Litim-type regulator shape function [107],

$$r(y) = \frac{1-y}{y}\mathbf{1}_{[0,1)}(y), \quad (28)$$

in which $\mathbf{1}_{[0,1)}$ denotes the characteristic function on the semi-open interval $[0,1) \subset \mathbb{R}$. This choice transforms all threshold functions (26) occurring in Eq. (25) into the following:

$$\begin{aligned}
&\mathrm{t}^4_{(10)}\left(w'_*;\frac{\eta_*}{2}\right) = \frac{1}{2}\int_0^1 y\frac{2-\eta_*(1-y)}{1-(1-w'_*)y}\mathrm{d}y, \\
&\mathrm{t}^4_{(10)}\left(1;\frac{\eta_*}{2}\right) = \frac{1}{2}\left(1-\frac{\eta_*}{6}\right), \\
&\mathrm{t}^4_{(11)}\left(w'_*,(w'_*\mathscr{F})';\frac{\eta_*}{2}\right) \\
&= \frac{1}{2}\int_0^1 y^2\frac{2-\eta_*(1-y)}{\left(1-(1-w'_*)y\right)\left(1-\left(1-(w'_*\mathscr{F})'\right)y\right)}\mathrm{d}y. \quad (29)
\end{aligned}$$

Inserting the ansatz (27) into the reduced FFE (25) and expanding the RHS in powers of $\mathscr{F}$, we obtain a tower of equations for the generalized fixed-point couplings $u_{i,*}$ by comparison of coefficients, the first four of which are listed below:

$$\begin{aligned}
u_{0,*} &= \frac{6-\eta_*}{192\pi^2}, \\
\eta_* &= \frac{8u_{2,*}}{48\pi^2+u_{2,*}}, \\
u_{2,*} &= \frac{1}{2}(4+\eta_*)u_{2,*} + \frac{3}{320\pi^2}(10-\eta_*)u_{2,*}^2 - \frac{5}{512\pi^2}(8-\eta_*)u_{3,*}, \\
u_{3,*} &= -\frac{1}{48\pi^2}(12-\eta_*)u_{2,*}^3 + \frac{3}{4}(4+\eta_*)u_{3,*} + \frac{3}{80\pi^2}(10-\eta_*)u_{2,*}u_{3,*} - \frac{1}{64\pi^2}(8-\eta_*)u_{4,*}. \quad (30)
\end{aligned}$$

We observe that the vacuum energy $u_{0,*}$ is fully determined by the anomalous dimension and completely decouples from the higher-order couplings. Having fixed the wave function renormalization such that $u_{1,*} = 1$, the anomalous dimension is fully defined in terms of $u_{2,*}$, such that the equations for the higher-order couplings can structurally be written as $u_{i,*} = f_i(u_{2,*}, \ldots, u_{i+1,*})$ for $i \geq 2$. The function f_i here features a linear dependence on the highest-order coupling $u_{i+1,*}$ involved.

This structure illustrates the role of initial conditions: since the FFE is a second-order ODE, two initial conditions are needed for specifying a solution. One initial condition is given by $u_{1,*} = 1$ through our choice of the wave function renormalization. A solution of the tower of equations, say up to order $N \in \mathbb{N}_0$, now requires one further initial condition. In principle, any coupling $u_{2,*}, \ldots, u_{N+1,*}$ could be fixed for this purpose. In order to construct a systematic expansion scheme, the standard strategy is to fix $u_{N+1,*}$ to some value, typically $u_{N+1,*} = 0$, and then increase N until some convergence criterion is met.

On the level of the FFE, this strategy corresponds to the initial conditions $w'_*(0) = 1$ and $w_*^{(N+1)}(0) = 0$. We call these initial conditions *improper*, because (i) they do not guarantee a unique solution since the conditions of the Picard-Lindelöf theorem are not matched, and (ii) they do not cover the full space of possible initial conditions and thus of the solution space.

In order to make this more precise, we define the partial sum $w_*(\mathscr{F}; N) \coloneqq \sum_{i=0}^{N} u_{i,*}\mathscr{F}^i$ as a truncation at order $N \in \mathbb{N}_0$. We can solve the resulting tower of equations (30), expressing all couplings as functions of $u_{2,*}$, i.e., $u_{i,*} = u_{i,*}(u_{2,*})$ for all $i \in \mathbb{N}_{0,\leq N}$. The improper initial condition $u_{N+1,*}(u_{2,*}) = 0$ yields a polynomial equation of degree $2N - 1$ in $u_{2,*}$ for $N \geq 1$. Therefore, it can at most admit $2N - 1$ zeroes. For each odd N, we find $N(\leq 2N - 1)$ distinct real solutions, whereas for each even N we instead obtain $N + 1(\leq 2N - 1)$ such solutions. The case $N = 0$ corresponds to $\eta_*(u_{2,*}) = 0$ and immediately implies $u_{2,*} = 0$ which describes the trivial solution. The solutions up to order $N = 26$ are shown in Fig. 1 (except for one solution existing only for even N which we consider as an artifact).

In addition to the noninteracting Gaussian fixed point (GFP) characterized by vanishing fixed-point couplings, we observe non-Gaussian fixed points (NGFP) which can be classified according to their number of RG relevant directions. The latter are characterized by the critical exponents $\Theta_j^{(N)}$ derived from the spectrum of the truncated stability matrix $B_*^{(N)}$, being the Jacobian of the (column) vector of beta functions, $\beta^{(N)} = (\beta_0, \beta_2, \beta_3, \ldots, \beta_N)^\mathsf{T}$ where $\beta_i \coloneqq \partial_t u_i(k)$ [note that $\beta_1 = \partial_t u_1(k) = 0$], with respect to the (column) vector of couplings $u^{(N)} \coloneqq (u_0, u_2, u_3, \ldots, u_N)^\mathsf{T}$ evaluated at the fixed-point candidate $u_*^{(N)}$,

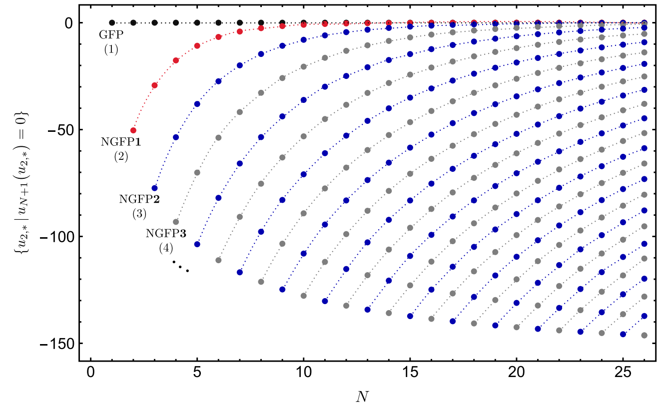

FIG. 1. Fixed-point candidates within the small-field expansion using improper initial conditions, $u_{N+1,*}(u_{2,*}) = 0$, as a function of truncation order $N = 1, \ldots, 26$. The dotted lines connect fixed-point candidates with the same number of relevant directions as labeled in parentheses below the fixed-point designations: GFP, NGFP**1**, etc. In addition to the Gaussian fixed point (black), we observe a fixed-point candidate with one relevant direction (red) and subsequent higher-order candidates (blue and gray), each of which moves toward weaker fixed-point couplings for increasing truncation order.

$$B_*^{(N)} = (D\beta^{(N)})(u_*^{(N)}), \quad -\Theta_j^{(N)} \in \mathrm{eig}(B_*^{(N)}). \tag{31}$$

Positive $\Theta_j^{(N)}$ mark RG relevant directions, corresponding to perturbations of a fixed point that grow large towards the IR and dominate the long-range physics. The number of relevant directions near a fixed point corresponds to the number of physical parameters to be fixed for predicting all low-energy observables. The increasing number of fixed-point candidates for increasing truncation order N also exhibit an increasing number of relevant directions. In Fig. 1, we have connected all fixed-point candidates with the same number of relevant directions by dotted lines to guide the eye, and labeled the non-Gaussian fixed points with the number of relevant directions in parentheses.

For the GFP, the only relevant direction is associated with the vacuum energy u_0. For the interacting fixed points, further nontrivial relevant directions appear in addition to that of the vacuum energy which remains exactly at $\Theta_0^{(N)} \equiv \Theta_0 = 4$ for all $N \in \mathbb{N}_0$, reflecting its canonical dimension. Classifying the non-Gaussian fixed-point candidates according to the same number of relevant directions for increasing truncation order in terms of fixed-point classes labeled by NGFP**n** as in Fig. 1, we observe that each class is characterized by $n + 1$ relevant directions, i.e. n nontrivial directions apart from the one of the vacuum energy.

If real, each of the NGFP**n** could represent a new universality class of nonlinear electrodynamics giving rise to UV complete quantum field theories of interacting light. However, following the standard reasoning in the literature for small-field expansions, we expect only those NGFP

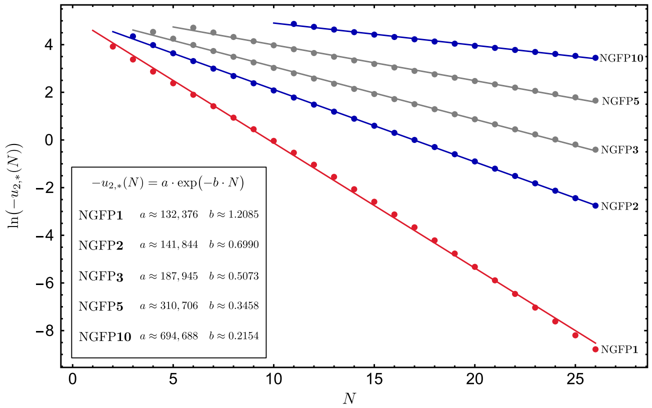

FIG. 2. Logarithmic plot of the (negative) fixed-point coupling $-u_{2,*}(N)$ for selected interacting fixed point classes NGFP**n** for increasing truncation order N. The results from the truncated fixed-point equations can be fitted to an exponential with fit parameters a, b such that $-u_{2,*}(N) = a\mathrm{e}^{-bN}$ (solid lines). The fit parameters are listed in the legend.

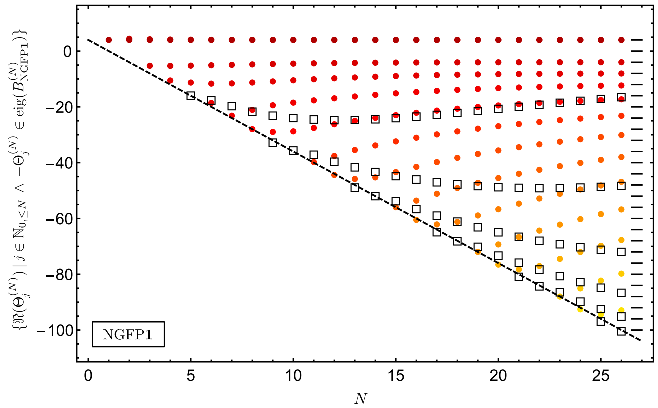

FIG. 3. Real parts ($\Re$) of critical exponents of the first non-Gaussian fixed point (NGFP**1**) derived from the small-field expansion as a function of truncation order N. Filled circles of equal color correspond to evolutions of individual critical exponents for increasing N. Open squares mark the real part of critical exponents which show up in complex conjugate pairs. The black dashed line follows the canonical mass dimension of the highest operator $\bar{\mathcal{F}}^N$ included at each N. Short solid ticks at the right edge mark the (negative of the) canonical mass dimensions of all operators occurring in the polynomial expansion, $4, 0, -4, -8, \ldots, -100$.

candidates to approximate a true interacting fixed point for which the couplings $u_{i,*}$ and the critical exponents converge with increasing truncation order N.

From Fig. 1, all fixed-point candidates NGFP**n** appear to converge towards the GFP. In order to substantiate this quantitatively, we plot the fixed-point value $\ln(-u_{2,*})$ as a function of truncation order N for a sample of the NGFPs in Fig. 2. We observe that the data follows linear fits, implying an exponential convergence with N: $u_{2,*}(N) = -a\exp(-bN)$ with fit constants a, b for each non-Gaussian fixed-point class NGFP**n**, respectively. Assuming that this exponential drop-off can be extrapolated to any N, we observe that $u_{2,*}$ and presumably also the higher-order couplings deplete to zero. Our data suggests that this also holds for higher-order NGFP**n** albeit with smaller damping rate. While this seems to suggest that all NGFP**n** converge to the GFP, we emphasize that we are dealing with different universality classes here, since the NGFP**n** exhibit a different number of relevant directions.

In order to check for the convergence of the critical exponents, we concentrate on NGFP**1** (marked in red in Fig. 1) which exhibits one nontrivial relevant direction in addition to the trivial vacuum-energy direction. For this purpose, we track the evolution of $B_{\mathrm{NGFP1}}^{(N)}$ and its spectrum, $\mathrm{eig}(B_{\mathrm{NGFP1}}^{(N)})$, for growing N. The result is depicted in Fig. 3. In addition to the critical exponent corresponding to u_0 which stays fixed at $\Theta_0^{(N)} = 4$ for all $N \in \mathbb{N}_0$, the second relevant exponent lies also close to the same value $\Theta_2^{(N)} \approx 4$; e.g., for our highest truncation, we find $\Theta_2^{(N=26)} - 4 < 10^{-5}$. The real parts of subsequent critical exponents remain negative (RG irrelevant) for all values of N studied here, and exhibit a clear tendency to approach the canonical mass dimensions of the higher-order operators, cf. black ticks in Fig. 3, right side. In particular, the convergence towards this asymptotic limit is already apparent for the first few irrelevant exponents. In addition, we also find complex conjugate pairs of critical exponents, the real parts of which are indicated by open squares. In the N range analyzed here, these complex pairs do not yet exhibit a clear signature of convergence, as is also true for the highest-order exponents. More definite answers would require higher truncations.

Our findings so far show a close similarity to those of [63] for nonlinear ED using a different truncation scheme as well as to those of [63,64] studying shift-symmetric scalar field theories both motivated by explorations of the weak-gravity bound [36,108–115]. In fact, the resulting FFEs in these systems show a great deal of similarity such that qualitative and even quantitative resemblance does not come as a surprise.

In the light of this similarity, we expect that also the further results of [64] for the shift-symmetric scalar field are also of relevance for nonlinear ED: for the eigenperturbations around the Gaussian fixed point of the scalar theory, the corresponding differential equation can be brought into Sturm-Liouville form which comes with an integration measure. However, the eigenperturbations around fixed points analogous to our NGFP**n** turn out not to be square-integrable with respect to the Sturm-Liouville measure. Reference [64] concludes for the shift-symmetric scalar theory that these fixed points do not represent legitimate physical fixed-point solutions and should be discarded, cf. [116–121].

Based on the strong similarity, we conjecture that an analogous Sturm-Liouville analysis leads to the same verdict for the fixed points NGFP**n** derived here from the small-field expansion using improper initial conditions. Therefore, the only trustworthy fixed point so far is the trivial Gaussian one with the field-strength potential $w_{\mathrm{GFP}}(\mathscr{F}) = \frac{1}{32\pi^2} + \mathscr{F}$.

This seems to suggest that no nontrivial fixed points exist in nonlinear ED; however, the argument is incomplete: as we argue in the following, the small-field expansion using improper initial conditions can be blind to further solutions. In order to illustrate this already within the small-field expansion, let us take a look at the behavior of the anomalous dimension η_* as a function of $u_{2,*}$. As discussed above, this relation is fully determined by the $\mathscr{F}$-linear part of the reduced FFE and can be found as the second equation from above in (30). In fact, this relation also holds for more general Taylor expansions of $w_*(\mathscr{F})$ around $\mathscr{F} = 0$, where $u_{2,*}$ has to be replaced by $\frac{1}{2}w_*''(0)$. A plot of η_* as a function of $u_{2,*}$ is shown in Fig. 4. For values $u_{2,*} \in (-\infty, -48\pi^2)$, the anomalous dimension is positive but large and reaches its limit point $\eta_* = 8$ for $u_{2,*} \to -\infty$. It moreover develops a pole at $u_{2,*} = -48\pi^2$, where beyond that pole in the range $u_{2,*} \in (-48\pi^2, 0]$ the anomalous dimension assumes negative values and crosses the zero point for vanishing $u_{2,*}$. In the opposite half space of positive $u_{2,*}$, η_* is smooth throughout, slowly monotonically growing and bounded from above by the limit value $\eta_* = 8$ for $u_{2,*} \to \infty$.

All non-Gaussian fixed points within the NGFP**n** displayed in Fig. 1 are located in the slim blue shaded region between the Gaussian solution and the pole. They cover thus a limited range of negative values for η_*. The shifted branch of extra solutions existing only for even N and essentially ignored in the discussion above corresponds to the separated blue shaded region at large $|u_{2,*}|$ beyond the pole. In fact, classifying this shifted branch as an artifact is also justified by the fact that the anomalous dimension is large, $\eta_* > 8$. Since our ansatz for the action is based on a derivative expansion, we expect the anomalous dimension not to exceed values of $O(1)$ as a self-consistency criterion of the expansion.

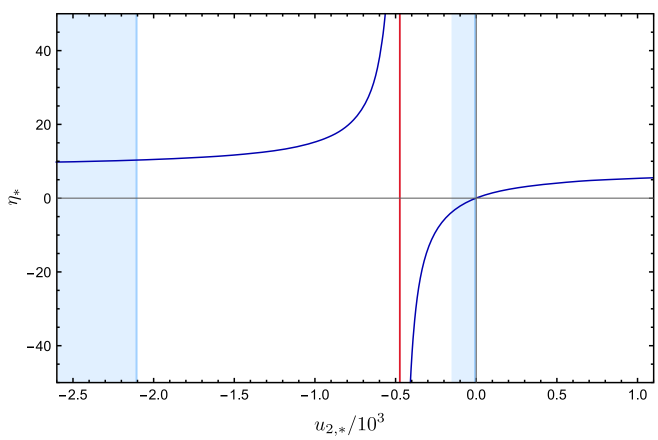

FIG. 4. Anomalous dimension η_* plotted as a function of the coefficient $u_{2,*}$ of the $\mathscr{F}^2$ contribution to the small-field expansion (blue line), exhibiting a pole at $u_{2,*} = -48\pi^2$ (red line). The blue shaded areas indicate the regions where all fixed points from the classes NGFP**n** based on the small-field expansion with improper initial conditions for $N \in \mathbb{N}_{\leq 26}$ have been found; the blue shaded segment at the left margin contains all fixed-point candidates classified as an artifact of the truncation, whereas the region right to the pole comprises the fixed-point candidates displayed in Fig. 1.

Most importantly, we observe that the small-field expansion together with the improper initial condition does not give access to solutions with small positive values of η_*. From the viewpoint of *proper initial conditions*, this appears to be unnatural: a proper initial condition for the FFE given, e.g., in terms of $w_*''(0) = u_{2,*}$ does naturally include small positive values of $u_{2,*}$ implying likewise small positive values of η_*. Whether or not such initial conditions lead to a legitimate fixed point and a global solution of the FFE needs to be and is studied separately in the following sections.

Let us conclude this section with a few comments on the limitations of the small-field expansion: In general, we expect the small-field expansion (27) to have a finite radius of convergence (ROC). This radius typically does not cover the maximal domain on which a full solution w_* can be defined, but rather a bounded interval $\mathscr{F} \in [0, \mathscr{F}_{\mathrm{ROC}})$. In the literature, numerical shooting methods have frequently been used [46,47,52,118,122] to identify the initial condition for $u_{2,*}$ by that value that maximizes $\mathscr{F}_{\mathrm{ROC}}$; this is based on the argument that a true fixed-point solution should be globally defined. Since our current form of the FFE is not suitable for shooting, we refrain from using this method, but complement our approach by a large-field expansion below.

Finally, if we dropped assumption (A2) and reincluded $\mathscr{G}^2$ dependencies into w_*, new features could appear in the $\mathscr{F}$ part of theory space as the new couplings act nontrivially on the RG flow. Since there is no concept of (Hodge) duality for a scalar field, theories of nonlinear ED with fixed functions $(\mathscr{F}, \mathscr{G}^2) \mapsto w_*(\mathscr{F}, \mathscr{G}^2)$ may no longer be comparable to shift-symmetric scalar systems. This may in principle affect the implications that we have conjectured from the FFE based on the analogy to the shift-symmetric case. We will come back to this point in Sec. VI.

V. FIXED FUNCTIONS FOR PROPER INITIAL CONDITIONS

In this section, we continue to use the approximations (A1) and (A2), but now aim at solving the FFE using *proper initial conditions*: as the reduced FFE (25) is a second order ordinary differential equation, two initial conditions are required to single out a unique solution. As an example, consider initial conditions at zero field amplitude, $w_*'(0) = w_1$ and $w_*''(0) = w_2$ with constants

$w_1, w_2 \in \mathbb{R}$. As discussed above, $w_1 = 1$ is already fixed by our choice for the wave function renormalization. This leaves us with a solution space, being a one-parameter family $\{\mathscr{F} \mapsto w_*(\mathscr{F}; w_2) | w_2 \in \mathbb{R}\}$.

In the present case where $w_*''(0) = u_{2,*}$, we could use $w_2 = u_{2,*}$ for parametrizing this family. For reasons that become clear later, we use the inversion of the exact relation between $u_{2,*}$ and the anomalous dimension η_* in the second equation of (30),

$$u_{2,*}(\eta_*) = 48\pi^2 \frac{\eta_*}{8 - \eta_*} \quad \text{for } \eta_* \neq 8, \tag{32}$$

in order to navigate through the space of solutions by using $\eta_* \in \mathbb{R}$ excluding the value $\eta_* = 8$ where (32) diverges.

In order to single out the physical fixed-point solutions, another criterion is needed. In many models, this criterion is given by global existence. For instance, in Ising-type systems, a generic choice for w_2 yields a solution with a singularity at a finite field amplitude [43,44], and only a single value of w_2 (or a discrete set) corresponds to a solution which exists for any value of the amplitude.

The following subsections are devoted to a construction of such global solutions using the analytical tools of small- and large-field expansions.

A. Small-field expansion

Let us start again with the small-field expansion, now implementing proper initial conditions. For this, we use again the Taylor expansion (27), leading to the tower of Eqs. (30). The essential difference for a given value of η_* now is that we can use the ith equation $u_{i,*} = f_i(u_{2,*}, \ldots, u_{i+1,*})$ and solve it exactly for $u_{i+1,*}$. Here we note two aspects: firstly, the solution is unique, since f_i depends linearly on $u_{i+1,*}$ and secondly, it is stable against increments of N for every admissible value of η_*. The latter means that the functional dependence of $u_{i+1,*}$ on the anomalous dimension is unaffected from the order of truncation and, once determined explicitly, applies to arbitrary N (provided that $i \leq N$, otherwise $u_{i+1,*}$ does not yet exist). The explicit expressions for the first few coefficients $u_{i,*}$ including the vacuum energy $u_{0,*}$ read:

$$\begin{aligned}
u_{0,*} &= \frac{6 - \eta_*}{192\pi^2}, \\
u_{1,*} &= 1, \\
u_{2,*} &= 48\pi^2 \frac{\eta_*}{8 - \eta_*}, \\
u_{3,*} &= \frac{6144\pi^4}{25} \frac{\eta_*}{(8 - \eta_*)^3} (160 + 150\eta_* - 19\eta_*^2), \\
u_{4,*} &= \frac{49152\pi^6}{125} \frac{\eta_*}{(8 - \eta_*)^5} (102400 + 236800\eta_* \\
&\quad + 67520\eta_*^2 - 24520\eta_*^3 + 1563\eta_*^4).
\end{aligned} \tag{33}$$

From investigating also higher order couplings we can find a general pattern, according to which the ith coupling for $i \geq 2$ can be written as a function of η_* as

$$u_{i,*}(\eta_*) = A_i \pi^{2(i-1)} \frac{\eta_*}{(8 - \eta_*)^{2i-3}} P_{2(i-2)}(\eta_*), \tag{34}$$

where A_i is a number and P_D denotes a full polynomial of degree D in η_*.

For illustration, let us study some explicit results for various choices of η_*. Since the improper initial conditions gave us access to negative values of η_* only which we argued to correspond to artifacts of the approximation, we now concentrate on the branch $\eta_* > 0$. Indeed, the fact that we are now capable of inspecting fixed points at positive anomalous dimensions is a notable difference between improper and proper initial conditions. As our truncation corresponds to a derivative expansion, we expect our approximation to be justified for small values of $\eta_* \lesssim O(1)$.

Let $w_*(\mathscr{F}; \eta_*, N)$ denote the Nth partial sum of Eq. (27). Several resulting field-strength potentials $\mathscr{F} \mapsto w_*(\mathscr{F}; \eta_*, N)$ in the range $\mathscr{F} \in [0, 0.02]$, for the choices $\eta_* \in \{10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}, 1, 5\}$ and $N \in \{10, 20, 30, 40\}$ are shown in Fig. 5. We observe that w_* describes a monotonically increasing function with a linear domain close to the origin, where approximately $w_*(\mathscr{F}; \eta_*, N) - u_{0,*}(\eta_*) \approx \mathscr{F}$. The range of this domain depends sensitively on the two parameters η_* and N. For example, for increasing N at fixed η_*, the point of departure from the linear behavior is shifted to smaller $\mathscr{F}$. However, the speed of this shift slows down rapidly for increasing N. Also for increasing η_* at fixed N, we observe a similar, if not more pronounced, effect.

Analogously to many other FRG studies of fixed-point potentials, we indeed expect the small field expansion to exhibit a finite radius of convergence; the preceding observed behavior is indicative for this. More quantitatively, let $r(\eta_*)$ denote the radius of convergence of the full power series (27), our observations suggest that $r(\eta_*)$ shrinks with increasing η_*. From the polynomial relation $u_{i,*}(\eta_*) \propto P_{2i-4}(\eta_*)$, it is clear that $\lim_{\eta_* \to \infty} r(\eta_*) = 0$, whereas $r(0) = \infty$, since $\eta_* = 0$ corresponds to the Gaussian fixed point, where all $u_{i\geq 2,*}$ vanish identically.

In order to compute the radius of convergence in full generality we would need to use Cauchy-Hadamard's theorem. Here, we confine ourselves to use a special case of the theorem where r can be extracted from the ratio test if all couplings are known, i.e., $r(\eta_*) = \lim_{i\to\infty} |u_{i,*}(\eta_*)/u_{i+1,*}(\eta_*)|$ whenever the limit exists, provided that all coefficients $u_{i,*}$ do not vanish above a certain index. The latter requirement is certainly fulfilled in the case at hand and we find the general result of the form

$$\frac{u_{i,*}(\eta_*)}{u_{i+1,*}(\eta_*)} = \frac{1}{\pi^2} \frac{A_i}{A_{i+1}} (8 - \eta_*)^2 \frac{P_{2(i-2)}(\eta_*)}{P_{2(i-1)}(\eta_*)}, \tag{35}$$

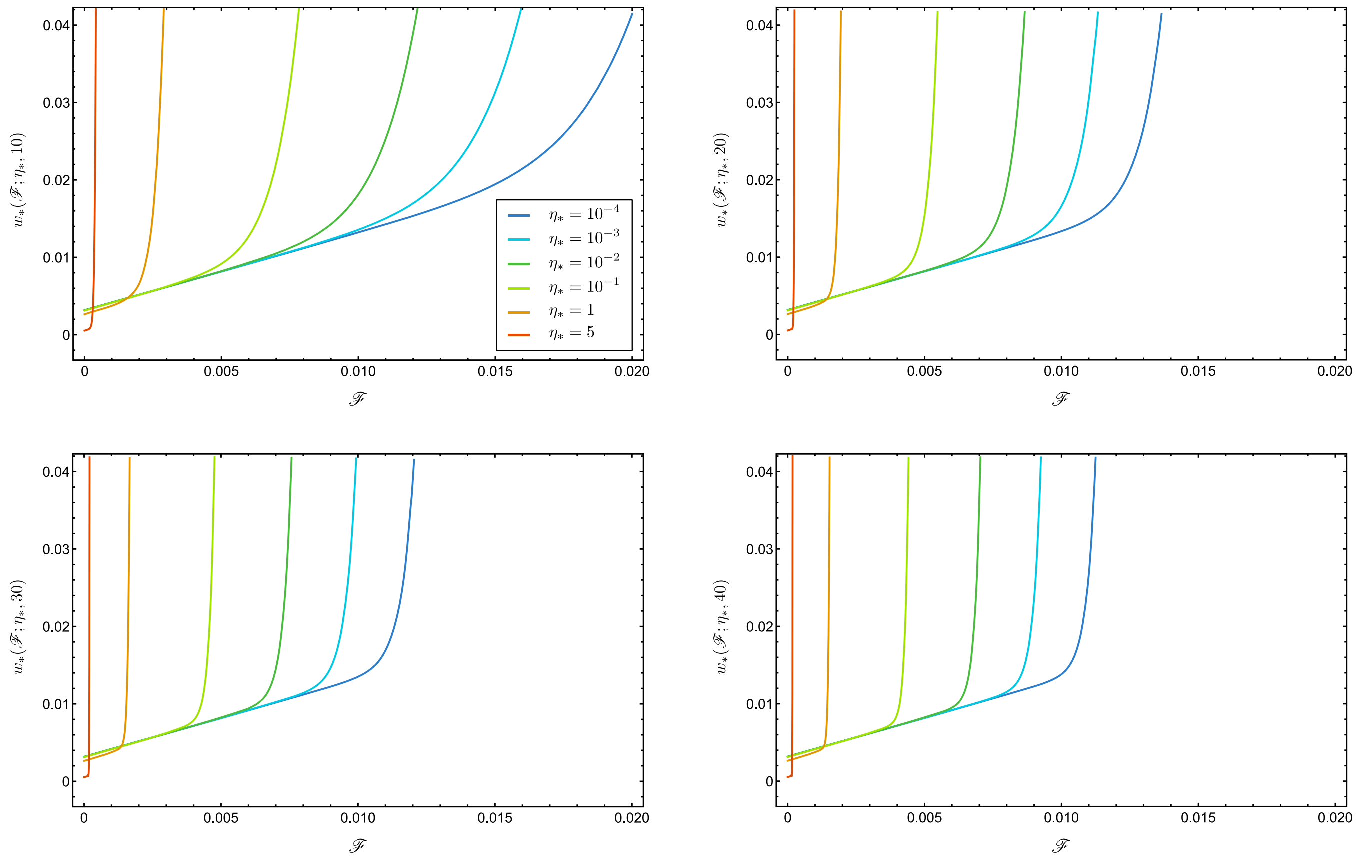

FIG. 5. (Partial) Fixed function w_* plotted as a function of positive $\mathscr{F}$ for different finite-dimensional truncations (N) and a selection of η_* values. The panes display results for increasing truncation order from $N = 10$ (upper left) to $N = 40$ (lower right) in steps of $\Delta N = 10$. In each panel, w_* is shown for values of η_* ranging logarithmically from 10^{-4} to 1 in colors from blue (right-most) to orange, respectively, also including the extreme example $\eta_* = 5$ (red/left-most) for illustrative purposes.

in which $\frac{A_i}{A_{i+1}} < 1$ holds for all i studied in this work. The sequence (35) is depicted in Fig. 6 for the values of η_* also considered in Fig. 5. Since each of these sequences exhibits rapid convergence, we obtain estimates for $r(\eta_*)$, e.g., $r(10^{-4}) \approx 0.009$ or $r(1) \approx 0.001$ [and, as anticipated, $r(5) \approx 0$ for the extreme example of a large anomalous dimension $\eta_* = 5$].

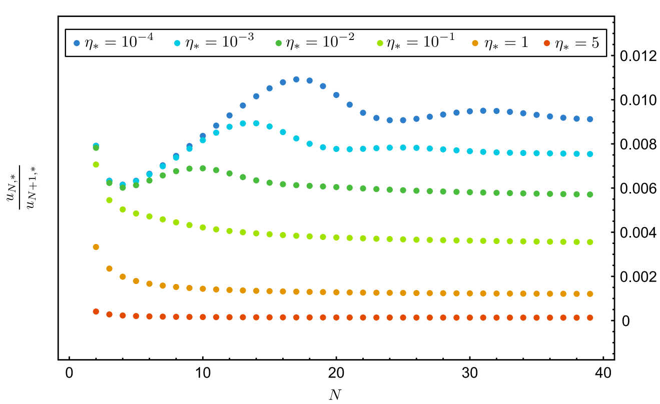

FIG. 6. Ratio sequence for the highest-order series coefficient $u_{N,*}$ of an Nth partial sum truncation of the small-field expansion Eq. (27). Here, the set of η_* values agrees with those of Fig. 5 following the same color code.

B. Large-field expansion

On our way to construct global solutions to the FFE, let us next study a large-field expansion. In order to identify a starting point, we use the following line of argument: On the one hand, we expect the field-strength potential to be an increasing function of the field amplitude. More specifically, we expect $w_*(\mathscr{F})$ to diverge for $\mathscr{F} \to \infty$, reflecting the fact that an infinitely large amplitude should cost an infinite amount of Euclidean action. On the other hand, the Sturm-Liouville analysis mentioned above suggests that the field-strength potential should be polynomially bounded. It is thus natural to assume that the field-strength potential diverges like a power for large amplitude, $w_*(\mathscr{F}) \sim \mathscr{F}^{\Delta}$ for $\mathscr{F} \to \infty$, with a positive exponent $\Delta > 0$. If so, both terms on the left-hand side of Eq. (25) scale like $\sim\mathscr{F}^{\Delta}$ to infinity at large fields. On the right-hand side, we observe that all terms are bounded: for $\Delta > 1$, all field-dependent terms are suppressed $\sim 1/\mathscr{F}^{\Delta-1}$; for $0 < \Delta \leq 1$, the right-hand side approaches a constant.

Ignoring subleading constants, the reduced FFE (25) therefore takes the asymptotic form

$$w_* - \left(1 + \frac{\eta_*}{4}\right) w'_* \mathscr{F} \sim 0 \quad (\mathscr{F} \to \infty). \tag{36}$$

Equation (36) corresponds to a first-order ordinary differential equation which can easily be solved analytically, yielding a two-parameter family of solutions with one integration parameter μ in addition to the anomalous dimension η_* parametrizing the proper initial condition of the full equation:

$$\begin{aligned} w_*(\mathscr{F}; \mu, \eta_*) &\sim \mu \mathscr{F}^{\Delta(\eta_*)} \quad (\mathscr{F} \to \infty), \\ \Delta(\eta_*) &:= \frac{4}{4 + \eta_*}. \end{aligned} \tag{37}$$

This demonstrates that our assumption of a power-law ansatz for the large-field asymptotics is self-consistent as long as $\eta_* > -4$. Moreover, we observe that this asymptotics is governed by the anomalous dimension. This is in complete analogy to many other examples in the literature where the large-amplitude asymptotics is balanced by the classical rescaling terms [i.e., the second term in Eq. (36)]. For the construction of the global solution below, the parameter μ will be fixed by the requirement of merging the small- and large-field solutions.

Using Eq. (37) as a leading order, we now need an ansatz for a systematic large-field expansion. For better readability, let us define $\mathscr{X} := 1/\mathscr{F}$ and rewrite the field-strength potential in terms of $\mathscr{X}$, $\tilde{w}_*(\mathscr{X}; \mu, \Delta(\eta_*)) = w_*(\mathscr{F}; \mu, \eta_*)|_{\mathscr{F} = \mathscr{X}^{-1}}$. Then, we may parametrize $\tilde{w}_*$ for small $\mathscr{X}$ as

$$\tilde{w}_*(\mathscr{X}; \mu, \Delta) = c(\mu, \Delta) + \mu \mathscr{X}^{-\Delta} + \tilde{\mathcal{C}}_*(\mathscr{X}; \Delta), \tag{38}$$

where c is a constant for a fixed pair (μ, Δ), and $\tilde{\mathcal{C}}_*$ provides the subleading terms of higher orders in $\mathscr{X}$. It is straightforward to check, that a naive power-series ansatz for $\tilde{\mathcal{C}}_*$ in general leads to artificial divergencies upon Taylor-expanding the threshold functions. Therefore a more refined strategy is needed.

The field-dependent part of the integrands of the involved threshold functions given in (29) can be written as either

$$\frac{1}{1 - Ay} \quad \text{or} \quad \frac{1}{1 - Ay} \cdot \frac{1}{1 - By}, \tag{39}$$

where A and B are $\mathscr{X}$-dependent quantities and $y \in [0, 1]$ is the variable of integration. The expressions in (39) can be expanded into a geometric series as long as $|A|, |B| < 1$. Given the ansatz (38) for $\tilde{w}_*$, these last-mentioned conditions can be viewed as restrictive boundary conditions on the applicability domain of $\mathscr{X}$, the range of which will be μ and Δ dependent. Using the explicit forms of A, B in Eq. (29), these conditions read

$$\begin{aligned} &0 < \Delta \mu \mathscr{X}^{1-\Delta} - \mathscr{X}^2 \tilde{\mathcal{C}}'_*(\mathscr{X}; \Delta) < 2, \\ &0 < \Delta^2 \mu \mathscr{X}^{1-\Delta} + \mathscr{X}^2 (1 + \mathscr{X} \tilde{\mathcal{C}}''_*(\mathscr{X}; \Delta)) < 2. \end{aligned} \tag{40}$$

Here, a prime denotes the derivative with respect to the argument, e.g., $\tilde{\mathcal{C}}'_*(\mathscr{X}; \Delta) \equiv \frac{\mathrm{d}\tilde{\mathcal{C}}_*(\mathscr{X}; \Delta)}{\mathrm{d}\mathscr{X}}$. Since $\tilde{\mathcal{C}}_*$ decreases fast enough for $\mathscr{X} \to 0$ by assumption, these conditions can be fulfilled if $1 - \Delta > 0$, such that contributions proportional to $\mathscr{X}^{1-\Delta}$ do not become arbitrarily large. It is interesting to note that this implies, in particular, $\eta_* > 0$, which is in fact the regime of interest to us.

The geometric series expansions of (39) on the one hand produce simple integrals over positive powers of y that can be performed analytically at any order. On the other hand, the resulting field dependencies arise from the corresponding powers A^n or B^n ($n \in \mathbb{N}_0$) which also contain powers of $\tilde{w}_*$ and its derivatives. This produces monomials $\propto \mathscr{X}^{m\Delta}$ ($m \in \mathbb{N}$) which cannot be covered by an ordinary power-series ansatz for $\tilde{\mathcal{C}}_*$. It rather needs to be expressed in terms of a formal Hahn series with $\Gamma(\Delta) \subset \mathbb{N}^2$ being a suitable Δ-dependent ordered group from which the running index of the corresponding sum is taken. Our final large-field ansatz results from an iterative process that covers all powers of $\mathscr{X}$ arising from the geometric-series expansion:

$$\begin{aligned} \tilde{w}_*(\mathscr{X}; \mu, \Delta) &= \sum_{e \in \Gamma(\Delta)} v_e(\mu, \Delta) \mathscr{X}^{p(e)} \\ &= c(\mu, \Delta) + \mu \mathscr{X}^{-\Delta} \\ &\quad + \sum_{I=1}^{\infty} \sum_{a=1}^{I} v_I^a(\mu, \Delta) \mathscr{X}^{I - a\Delta}. \end{aligned} \tag{41}$$

Here, v_I^a $((I, a) = e \in \Gamma(\Delta) \subset \mathbb{N}^2)$ are the μ- and Δ-dependent coefficients to be determined and $p(e) := I - a\Delta$.

Unlike in the small-field regime, a truncation for the large-field sector will thus be specified by two parameters $N_1, N_2 \in \mathbb{N}_0$. The former truncates the first series (41) to its first N_1 terms $I \in \mathbb{N}_{\leq N_1}$, including a total of $\frac{1}{2} N_1(N_1 + 1)$ terms because of the double sum. Besides the lowest order contribution, which is always $\mathscr{X}^{1-\Delta}$, this also defines the highest power of this expansion which is given by $\mathscr{X}^{N_1 - \Delta}$. Higher order contributions are neglected whenever they are generated by the expansion of the reduced FFE. However, this procedure is not fully self-consistent insofar as also contributions proportional to $\mathscr{X}^{p(e)}$ with $1 - \Delta < p(e) < N_1 - \Delta$ will emerge, but which are not part of our truncation itself. Since those p's can still be expressed as $p(e) = I - a\Delta$ for some pair $(I, a) = e \in \Gamma(\Delta)$, these terms eventually get successively and consistently resolved at higher truncations N_1. In order to estimate the quantitative impact of these terms, we will compare two different truncations and try to evaluate the weight of these additional contributions. The second parameter N_2 limits the geometric series emerging through (39) to contain only their first N_2 terms. The effect of choosing different N_2 values can be understood as follows: if we think of all possible contributions $\propto \mathscr{X}^{I-a\Delta}$ being classified by an ordered sequence of sets which contain all exponents according to their numerical value in the interval $I_n \coloneqq [n, n+1)$ for $n \in \mathbb{N}_0$, the parameter N_2 causes the number of terms which actually appear on the RHS of the reduced FFE from each of these classes to increase. For instance, if Δ is less than, but close to, one (which corresponds to small η_*), terms with exponents $I - a\Delta$ for $a = I$ give $I(1-\Delta) \in [1, 2) = I_1$ for many values of I. Given a fixed parameter value of N_1, we may generate more and more terms that belong to the class I_1 which are yet not part of the truncation at hand if N_2 gets large enough. In this sense, N_2 controls the resolution at which the spectrum of all potential contributions taken from the classes I_n and emerging on the RHS of the reduced FFE is sampled. However, only if we also raise N_1 we would be able to balance this effect with the LHS and reveal the information available at this level of the resolution. Thus, it is advisable to narrow the pertinent truncations on similar values, e.g., $N_1 = N_2 + 1$.

C. Global fixed functions

Let us now proceed with the construction of global fixed functions based on the analytic expansions for small and large fields employing proper initial conditions. Since both expansions generically have a finite radius of convergence, it is *a priori* unclear whether both expansions have a finite overlap region where they can be matched using the parameter μ. For instance, for scalar $\mathrm{O}(N)$ models, such an overlap region does exist. If so, it is typically not possible to perform the matching for any set of proper initial condition parameters. In fact for scalar models this is possible only for a discrete set of initial conditions that correspond to a discrete set of fixed points [123], such as the Gaussian or the Wilson-Fisher fixed point. If this standard scenario applies to the present case, we should expect that it singles out specific values of η_* for which global fixed functions can be constructed. Incidentally, there is *a priori* no guarantee that a finite overlap region for the two expansions exists; see [59] for a counter example. In this case, the present approach would not find a viable global solution and more powerful methods such as those of [48,59,124,125] are needed.

Let us now construct estimates for global fixed functions obeying proper initial conditions using the following steps:

(1) First, we construct a solution w_*^{L} from the large-field expansion of the reduced FFE defined in terms of the two truncation parameters $N_1 \in \mathbb{N}, N_2 \in \mathbb{N}_0$. The ansatz (41) together with the reduced FFE yields an algebraic system from which we can determine the constant c as well as the $\frac{1}{2}N_1(N_1+1)$ unknown coefficients $v_I^a \in \{v_1^1, v_2^1, v_2^2, v_3^1, \ldots, v_{N_1}^{N_1}\}$. The latter are derived as functions of μ and Δ.

(2) Second, we construct the small-field expansion w_*^{S} for proper initial conditions specified in terms of a value for η_* based on the highest-order truncation, i.e., the largest value of N used in Sec. V A. The proper initial condition, of course also fixes the $\Delta = \Delta(\eta_*)$ dependence of the coefficients v_I^a, and thus that of the large-field solution w_*^{L} from step (1) which retains only a μ dependence.

(3) In order to specify the remaining free parameter μ, let us first quantify the overlap region of the two approximate solutions w_*^{S} and w_*^{L}: from our construction of w_*^{S}, we also obtain an approximation of the radius of convergence r, cf. Eq. (35) and Fig. 6. Unfortunately, a formal Hahn series like (41) does generally not allow for a proper notion of convergence, let alone a radius of convergence; still, we observe that w_*^{L} develops a pronounced barrier b_{L} below which the derivative $(w_*^{\mathrm{L}})'$ rapidly increases. This happens for sufficiently small $\mathscr{F}$ numerically comparable to some power of the coefficients v_I^a. A numerical value for b_{L} can be estimated by $b_{\mathrm{L}} \approx \max\{(v_I^a)^{\frac{1}{I-a\Delta}} | I \in \mathbb{N}_{\leq N_1}, a \leq I\}$. We use b_{L} as a provisional substitute for a radius of convergence for w_*^{L}.

Then, we define the overlap region of w_*^{S} and w_*^{L} in terms of the interval intersection $[0, r] \cap [b_{\mathrm{L}}, \infty) = [b_{\mathrm{L}}, r]$ if $b_{\mathrm{L}} \leq r$. (If $b_{\mathrm{L}} > r$, there is no overlap region and a construction of a global solution cannot be based on the small- and large-field expansions alone.) Even though b_{L} carries a μ dependence in principle, we observe this dependence to be rather weak; in practice, the approximation $b_{\mathrm{L}} \approx \mathrm{const.}$ can hence be used for a wide range of μ values.

Now, we fix the free parameter μ by demanding that the square-deviation integral in the overlap region,

$$\delta^2(\mu) \coloneqq \int_{[b_{\mathrm{L}}, r]} (w_*^{\mathrm{S}}(\mathscr{F}) - w_*^{\mathrm{L}}(\mathscr{F}; \mu))^2 \mathrm{d}\mathscr{F}, \quad (42)$$

becomes minimal,

$$\mu \to \mu_0 \colon \ (\delta^2)'(\mu_0) = 0 \wedge (\delta^2)''(\mu_0) > 0. \quad (43)$$

In case of several local minima, we pick the μ_0 value for the global minimum of δ^2. Inserting μ_0 into w_*^{L} finally completes the large-field solution.

(4) As a last step, both w_*^{S} and w_*^{L} are suitably glued within the interval $[b_{\mathrm{L}}, r]$. A simple procedure is to

use an intersection point of the two expansions. If more than one intersections appear, say $\mathscr{F}_1, \mathscr{F}_2, \ldots$, we choose the one for which the absolute difference $|(w_*^{\mathrm{S}})'(\mathscr{F}_\ell) - (w_*^{\mathrm{L}})'(\mathscr{F}_\ell;\mu_0)|$ is minimal to achieve the smoothest transition possible with this direct method. Suppose we have chosen such a point of intersection in this way, $\mathscr{F}_\ell$ for some index ℓ, then replace $\mathscr{F}_\ell \to \mathscr{F}_0$ and let $\mathscr{F}_0 \in [b_{\mathrm{L}}, r]$. With this choice, we obtain an approximate global solution from

$$w_*(\mathscr{F}) = (w_*^{\mathrm{S}} \cdot \mathbf{1}_{[0,\mathscr{F}_0)} + w_*^{\mathrm{L}} \cdot \mathbf{1}_{[\mathscr{F}_0,\infty)})(\mathscr{F}). \qquad (44)$$

The results for two different truncations $(N_1, N_2) \in \{(2,1),(4,3)\}$ are presented in Fig. 7.

Based on the quantitative results, we can first and foremost conclude that satisfactory approximations to continuous global fixed functions can be constructed for a variety of different η_* values and truncations. With our matching condition (43) we even obtain comparatively smooth fixed functions at the fusion point $\mathscr{F}_0$ for sufficiently small $\eta_* > 0$. For the case $\eta_* = 1$, it is evident from Fig. 7 that this gluing procedure leads to a visible kink at the matching point. This may already be taken as an indication that $\eta_* = 1$ lies beyond the range of η_* values for which the present procedure yields a valid approximation of a global fixed function. This is discussed in more detail in the next section, where also another gluing procedure will be presented that yields differentiable global approximations.

Let us finally comment on the convergence properties of our truncated expansion for $N, N_1, N_2 \to \infty$. For this, we use the dependence of the overlap region $[b_{\mathrm{L}}, r](N, N_1, N_2; \eta_*)$ on the truncation parameters and the anomalous dimension as an indicator. For fixed N and η_* we can infer from Fig. 7 that the interval obeys the inclusion relation $[b_{\mathrm{L}}, r](N, N_1, N_2; \eta_*) \supset [b_{\mathrm{L}}, r](N, N_1+m+1, N_2+m; \eta_*)$, for $m \in \mathbb{N}$. More generally, we expect that increasing the

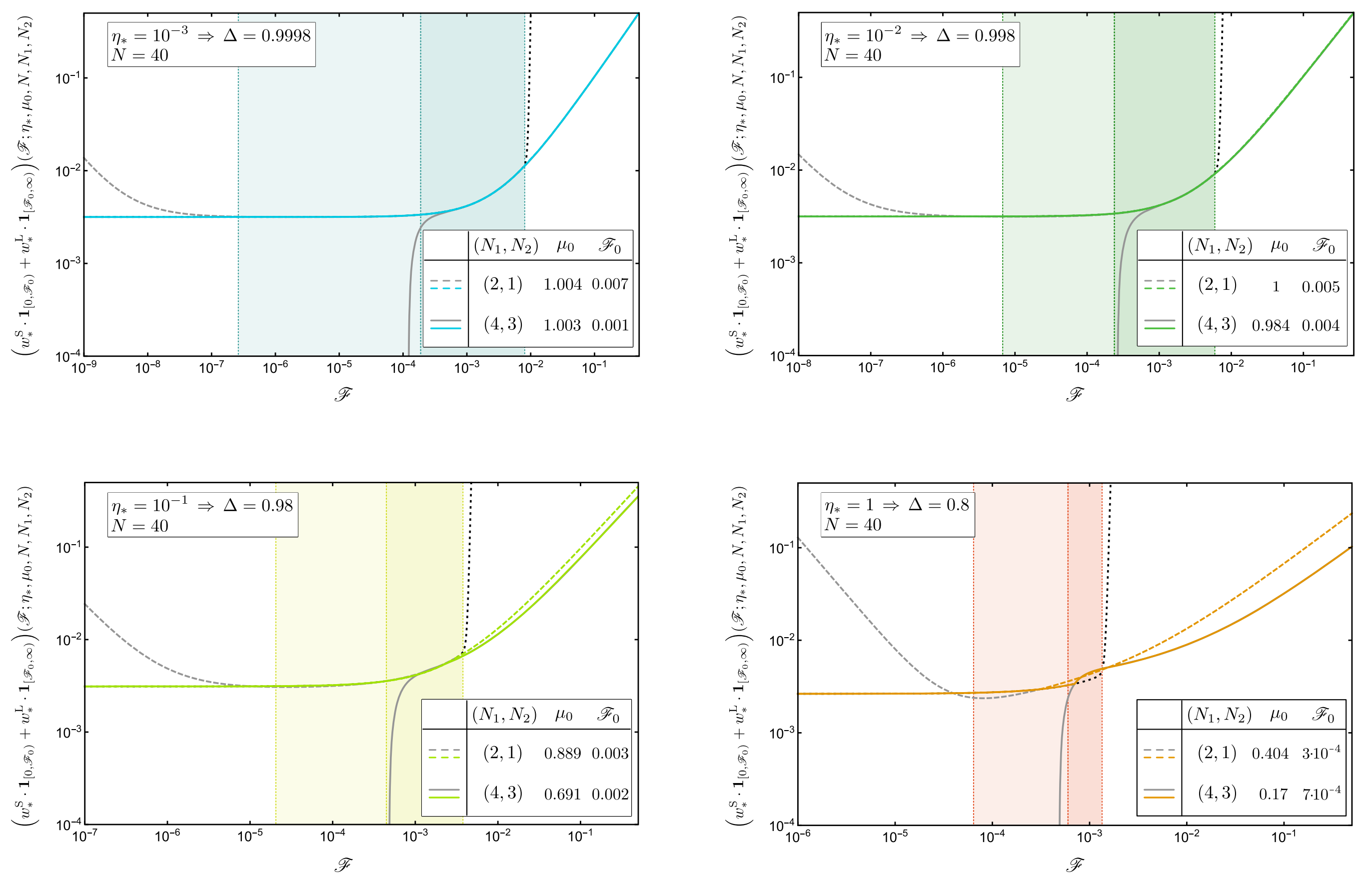

FIG. 7. Global continuous fixed functions and residuals of small- and large-field expansions for various values of the anomalous dimension η_* for different truncation orders. The color code for different values of η_* corresponds to that of Fig. 5. The large-field asymptotics is given by a power law with exponent $\Delta = \frac{4}{4+\eta_*}$, cf. upper left legends. The small-field expansion is truncated beyond $N = 40$, whereas the large-field truncation parameters (N_1, N_2) are taken to be either (2,1) (dashed) or (4,3) (solid). In each panel, the colored lines show combined global solutions according to Eq. (44), where the parameters μ_0 and $\mathscr{F}_0$ are such that (43) is fulfilled and the intersection has the least slope difference. Numerical values are given in the boxes in the lower right corners. Moreover, gray lines represent pure large-field solutions in both truncations, whereby the pure small-field solution is distinguished from them by the black dotted line in each panel. The overlap region, i.e., the interval $[b_{\mathrm{L}}, r]$ is indicated by the colored regions.

large-field truncation makes the overlap region smaller for fixed N and η_*. Increasing both N and η_* in addition amplifies this effect, because it reduces the radius of convergence r. Quantitatively, the overlap regions depend sensitively on N_1, N_2 and η_*. For instance, for $\eta_* = 1$ at $(N_1, N_2) = (4, 3)$ the overlap spans much less than one order of magnitude. For small anomalous dimensions, the overlap is considerably larger, but also contracts sizably when going from the (2,1) truncation to the (4,3) truncation. Still, the overlap remains sufficiently large to obtain a comparatively smooth global approximation in contrast to the $\eta_* = 1$ case.

Of course, if the small- and/or large-field expansion is only an asymptotic series, then the overlap region will eventually vanish for large truncation parameters. Nevertheless, finite truncations would then still represent quantitatively trustworthy approximations that serve to construct global solutions. We consider the approximations constructed in the present section to provide satisfactory evidence for the existence of a continuous family of solutions for small values of η_*.

D. Absence of a movable singularity

The existence of a continuous family of fixed functions for small positive η_* evidenced by our preceding construction is rather unusual. While the initial conditions, in principle, allow for a continuous solution family, the intrinsic nonlinearity of the FFE reduces this continuous set typically to a discrete set of solutions.

For instance for the paradigm example of scalar $\mathrm{O}(N)$ models, the matching of small- and large-field expansions imposes a condition that is only satisfied for a finite set of solutions (typically only one solution corresponding to the Wilson-Fisher fixed point). This can also be rephrased as follows: the proper initial conditions at small field amplitude can also be reformulated as boundary conditions to be imposed in the small- and large-field limit, e.g., in terms of the potential derivative at zero field and the large-field asymptotics. While an initial-value problem can feature continuous solution sets, a boundary-value problem can single out discrete solutions.

Another way to see this reduction or "quantization" of fixed-function solutions goes as follows: bringing the FFE to normal form, the differential equation, e.g., for the $\mathrm{O}(N)$ model reads

$$v''(\varphi) = \frac{e(v, v'; \varphi)}{s(v, v'; \varphi)}, \tag{45}$$

where v denotes the potential, φ the field amplitude, and e and s are functions of the potential and its first derivative. In particular, s typically corresponds to the scaling term, i.e., the $\mathrm{O}(N)$ analog of the left-hand side of the FFE (25). For generic initial conditions imposed at $\varphi = 0$, the denominator s develops a zero at some finite φ which is called a movable singularity of the FFE. If such a movable singularity exists, a global solution for $v(\varphi)$ can only be constructed provided that e also vanishes at this zero of s. This imposes another condition on the initial values and thus leads to a quantization of solutions [46,47,52,118,122].

For the present case, this implies that our continuous family of solutions to the FFE for small η_* persists only if the FFE (25) does not feature such a movable singularity. Unfortunately, this is difficult to check directly, since we cannot bring Eq. (25) analytically into normal form.

In order to collect indirect evidence, we proceed as follows: we first expand the integrands of the threshold functions in Eqs. (29) in a geometric series and perform the loop integration term by term. For instance, to lowest nontrivial order, we obtain the differential equation

$$a_0 + a_1 \cdot (\mathscr{F} w_*'') + a_2 \cdot (\mathscr{F} w_*'')^2 = 0, \tag{46}$$

with coefficients

$$\begin{aligned} a_0(\mathscr{F}, w_*'; \eta_*) &= 280 - 45\eta_* \\ &\quad + 5\left[\eta_* - 8 - 96\pi^2\left(1 - \frac{1}{\Delta(\eta_*)}\mathscr{F}\right)\right] w_*', \\ a_1(w_*'; \eta_*) &= (13\eta_* - 124) - 2(5\eta_* - 54) w_*' \\ &\quad + 2(\eta_* - 12) {w_*'}^2, \\ a_2(w_*'; \eta_*) &= (5\eta_* - 54) + 2(\eta_* - 12) w_*'. \end{aligned} \tag{47}$$

Equation (46) is a quadratic polynomial in $\mathscr{F} w_*''$ and can straightforwardly be brought into normal form analogously to Eq. (45). To this order, it turns out that the resulting condition for the absence of a movable singularity is satisfied if and only if $a_2 \neq 0$. Conversely, if a movable singularity is present, say at $\mathscr{F} = \mathscr{F}_{\mathrm{ms}}$, then we find

$$w_*'(\mathscr{F}_{\mathrm{ms}}; \eta_*) = -\frac{1}{2}\frac{54 - 5\eta_*}{12 - \eta_*}. \tag{48}$$

Since our attention is devoted to small $\eta_* > 0$, this expression signifies a negative slope of the field strength potential at the movable singularity. However, if the convergence criterion of the geometric series expansion of the threshold functions is fulfilled, that is if

$$\forall\, \mathscr{F} \in \mathbb{R}_0^+ :\ w_*'(\mathscr{F}), w_*'(\mathscr{F}) + \mathscr{F} w_*''(\mathscr{F}) \in [0, 2), \tag{49}$$

then Eq. (48) cannot be true for any $\mathscr{F}$ and thus there is no movable singularity.

In order to check this, we have to construct global solutions which are differentiable at least twice. For this, we use an interpolation of the small- and large-field expansions in the overlap region $[b_{\mathrm{L}}, r]$ by means of an affine combination:

$$w_* = g_{\mathrm{S}} w_*^{\mathrm{S}} + g_{\mathrm{L}} w_*^{\mathrm{L}} = w_*^{\mathrm{L}} + g_{\mathrm{S}}(w_*^{\mathrm{S}} - w_*^{\mathrm{L}}), \tag{50}$$

where we have used that the weight functions $g_{\mathrm{S}}, g_{\mathrm{L}}$ add up to unity; $g_{\mathrm{S}} + g_{\mathrm{L}} = 1$.

The interpolation via g_{S} is constructed in such a way that the contributions of the more reliable approximation dominates the derivatives near the edges of the overlap region, e.g., w_*^{S} dominates near b_{L} and w_*^{L} dominates near r. This construction therefore avoids artifacts and contaminations from the less trustworthy approximation in the derivatives. On the level of the field-strength potential, g_{S} gives full weight to the small-field expansion at $\mathscr{F} = b_{\mathrm{L}}$, but suppresses it completely at $\mathscr{F} = r$, i.e., $g_{\mathrm{S}}(b_{\mathrm{L}}) = 1$ and $g_{\mathrm{S}}(r) = 0$. This guarantees a seamless transition at the interval endpoints; $w_*(b_{\mathrm{L}}) = w_*^{\mathrm{S}}$ and $w_*(r) = w_*^{\mathrm{L}}$. For concreteness we consider the following one-parameter family of weights for any fixed overlap interval $[b_{\mathrm{L}}, r]$:

$$g_{\mathrm{S}}(\mathscr{F};\alpha) = \frac{1}{2}\left[1 - \frac{\tanh\left(\alpha\left(\frac{\mathscr{F}-b_{\mathrm{L}}}{r-b_{\mathrm{L}}} - \frac{1}{2}\right)\right)}{\tanh(\frac{\alpha}{2})}\right], \qquad (51)$$

with a continuous parameter $\alpha \in \mathbb{R}\backslash\{0\}$. By varying α we can control the profile of the weight function between the endpoints and regulate the transition sharpness from w_*^{S} to w_*^{L} near the midpoint at $\mathscr{F} = \frac{1}{2}(b_{\mathrm{L}} + r)$. Small values of α provide only slight deviations from the linear weight function, whereas larger values progressively pronounce the kink of the tanh graph. In order to avoid artifacts induced by the derivative of the weight function itself, which could, in particular, jeopardize the convergence criteria in (49), α should not be chosen excessively large. On the other hand, for reasonably large values of α, g_{S} becomes flat near the endpoints. In this way we can neglect derivatives of g_{S} in the corresponding region if α is not too small and confer derivatives of the field strength potential to any order a form similar to Eq. (50). This behavior is indeed suited for a smooth transit to the more reliable approximations beyond the overlap interval. Hence, the discussion suggests to find an adequate compromise between relatively flat ends and a gentle slope for $g_{\mathrm{S}}|_{[b_{\mathrm{L}},r]}$. Reasonable choices for α are usually of order one but can be varied by an order of magnitude.

Studying the convergence criteria (49), we have verified explicitly that our global solutions for $\eta_* = 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}$ satisfy these criteria for a wide range of parameter values α and thus are compatible with the absence of a movable singularity. In addition, we have verified that this statement also holds for the FFE to next order in the geometric-series expansion (the resulting FFE are too extensive to be written down explicitly here).

By contrast, the criterion for the convergence of the geometric series expansion is violated by the solution for $\eta_* = 1$, independently of α. While this may solely indicate a failure of this expansion, we take this as an indication that the FFE may possess a movable singularity for sufficiently large η_*. If so, further solutions may still exist for discrete values of η_*. However together with the fact that the directly glued solutions exhibit a kink and that the overlap region of the expansions is rather small, we consider the present observation as a further piece of evidence that $\eta_* = 1$ as well as larger values do not support a global solution to the FFE.

We conjecture that a continuous family of global fixed-function solutions exists for a finite interval $\eta_* \in (0, \eta_{\mathrm{cr}})$ where the critical anomalous dimension η_{cr} lies in between $1/10$ and 1.

E. Near critical regime

Having constructed a global fixed function for nonlinear electrodynamics in the truncated theory space, we now return to the analysis of the near critical region for our solutions found with proper initial conditions. Of central interest are the critical exponents of perturbations and the classification of (ir)relevant directions.

In principle, we would have to construct eigenperturbations of the global fixed function in order to reliably read off the eigenvalues of the stability matrix after insertion of the global solution. In view of the complexity of the FFE, we resort to a simpler method which we expect to give reasonable results for the leading-order exponents: we simply use the stability matrix arising from the small-field expansion inserting the fixed-point results for the coefficients $u_{i,*}$ that we obtain from the small-field expansion using proper initial conditions. As the latter can all be expressed as functions of η_*, the stability matrix $B(\eta_*) = (D\beta)(\eta)|_{\eta=\eta_*}$ becomes a function of only the anomalous dimension η_*.

Truncating the small-field expansion at order N, the stability matrix reduces to an $N \times N$ submatrix $B^{(N)}(\eta_*)$, the eigenvalues of which we can determine straightforwardly in order to obtain the critical exponents $\Theta_j^{(N)}$ $(j \in \mathbb{N}_0)$, cf., Eq. (31). The latter are thus computable as functions of η_* for increasing truncation N. As before, $\Theta_0^{(N)} \equiv \Theta_0 = 4$ reflecting the canonical dimension of the vacuum energy holds independently of the truncation. At low truncation orders, also the leading-order results for the critical exponents can be worked out analytically. It is instructive to take a look at the leading nontrivial exponent Θ_2 associated essentially with the coupling $u_{2,*}$. At order $N = 2$, we have

$$\Theta_2^{(2)}(\eta_*) = -\frac{640 + 1360\eta_* - 153\eta_*^2}{20(8 - \eta_*)}. \qquad (52)$$

In the limit $\eta_* \to 0$, we rediscover $\Theta_2^{(2)}(0) = -4$ equaling the canonical mass dimension of the dimensionful coupling $\bar{u}_2$ as it should. For small $\eta_* > 0$, the exponent receives small corrections. At higher truncation order, the critical exponent can pick up an imaginary part, so that we focus on the real part ($\Re$) in the following. For instance at truncation order $N = 3$, the expression $\Re[\Theta_2^{(3)}](\eta_*)$ is more extensive, but essentially of the form (52) replacing the quadratic polynomial in the numerator by a cubic plus the square root of a septic polynomial in η_* and the denominator by $40(8-\eta_*)^2$. A similar modification applies for the transitions from $\Re[\Theta_2^{(3)}]$ to $\Re[\Theta_2^{(4)}]$ and thereafter to $\Re[\Theta_2^{(5)}]$. Consequently, Eq. (52) admits a pole at $\eta_* = 8$, as expected: this pole must always be present for all critical exponents at each order N since all couplings diverge at that parameter value.

As another check, we can make contact with our results for the critical exponents using improper initial conditions, as studied in Sec. IV B. Therein, we found a positive critical exponent $\Theta_2^{(N)}$ for all truncations studied (with hindsight considered as an artifact of the improper initial conditions). This suggests that $\Theta_2^{(N)}$ should feature a zero as a function of η_*. In fact, for $\Theta_2^{(2)}$ we find a zero at $\eta_* \approx -0.448 =: \tilde{\eta}_*^{(2)}$, using Eq. (52). At $N = 2$, larger anomalous dimensions $\eta_* \in (\tilde{\eta}_*^{(2)}, 8)$, thus produce an irrelevant coupling $u_{2,*}(\eta_*)$. It becomes relevant for smaller values $\eta_* < \tilde{\eta}_*^{(2)}$ in agreement with our findings of Sec. IV B.

Moving to $N = 3$, we can essentially observe the same behavior for $\Re[\Theta_2^{(3)}]$ with the zero shifting to a larger value, $\eta_* \approx -0.229 =: \tilde{\eta}_*^{(3)}$. In addition, we find several regions where $\Theta_2^{(3)}$ switches from a real- to a complex-valued number, especially near the pole. Hence, $\Re[\Theta_2^{(3)}]$ has discontinuities at these points. Whenever $\Theta_2^{(3)}(\eta_*) \in \mathbb{C}$ in these regions, then, by the complex conjugate root theorem, also its complex conjugate $\overline{\Theta_2^{(3)}(\eta_*)}$ must be an eigenvalue. Now, because $\Theta_0^{(N)} = 4$ is true for all $N \in \mathbb{N}_0$, we must have $\overline{\Theta_2^{(3)}} = \Theta_3^{(3)}$, i.e., the critical exponents belonging to $u_{2,*}$ and $u_{3,*}$ must combine to complex conjugate pairs. The situation only marginally changes when we increase the truncation to $N = 4$ and $N = 5$. The switching behavior of $\Theta_2^{(4)}$ and $\Theta_2^{(5)}$ between $\mathbb{R}$ and $\mathbb{C}$ is unpredictably chaotic. On the other hand, the zeroes $\tilde{\eta}_*^{(4)} \approx -0.13$ and $\tilde{\eta}_*^{(5)} \approx -0.079$ where $u_{2,*}$ becomes relevant move closer to 0. For reasons of continuity, we do not expect that this sequence of zeroes, $(\tilde{\eta}_*^{(N)})_{N\in\mathbb{N}}$, crosses zero, where $\Theta_2^{(N)}(0) = -4$ must hold to all orders. The continuity assumption hence implies that $\tilde{\eta}_*^{(N)} < 0$ for all $N \in \mathbb{N}$. In this scenario, $u_{2,*}$ would be an irrelevant coupling for all positive η_*, at least sufficiently below the pole at $\eta_* = 8$. Whether this continuity scenario applies to all orders remains an open question.

For even larger $N \geq 6$, no elementary closed-form solutions to the characteristic polynomial of $B^{(N)}(\eta_*)$ for arbitrary η_* exist according to the Abel-Ruffini theorem. Therefore, we continue with specific η_* values as done in Figs. 5 and 6, and discuss some properties of the full spectra $\mathrm{eig}(B^{(N)}(\eta_*))$ up to $N = 26$.

Let us start with the first nontrivial critical exponents $\Theta_2^{(N)}$ and $\Theta_3^{(N)}$. Their η_* dependence and evolution for an increasing dimension of theory space N is presented in Fig. 8. For small η_*, both $u_{2,*}$ and $u_{3,*}$ describe irrelevant couplings and are numerically close to the canonical mass dimension of their respective dimensionful versions with minimal variation throughout various N. As η_* gets larger, both of the real parts approach each other, which is particularly noticeable for small N, before they eventually combine to a complex conjugate pair. Most importantly, in either case both real parts seem to converge in the large-N limit and thus indicate a well-defined critical structure for the fixed functions constructed from a small-field expansion up to the third-order operator $\mathcal{F}^3$.

Unfortunately, the small-field expansion technique used here for an estimate of the leading critical exponents fails for even higher exponents $\Theta_4^{(N)}, \Theta_5^{(N)}, \ldots$. Of course, for any finite N, estimates for the exponents are computable, but we do not observe any sign of convergence even for large values

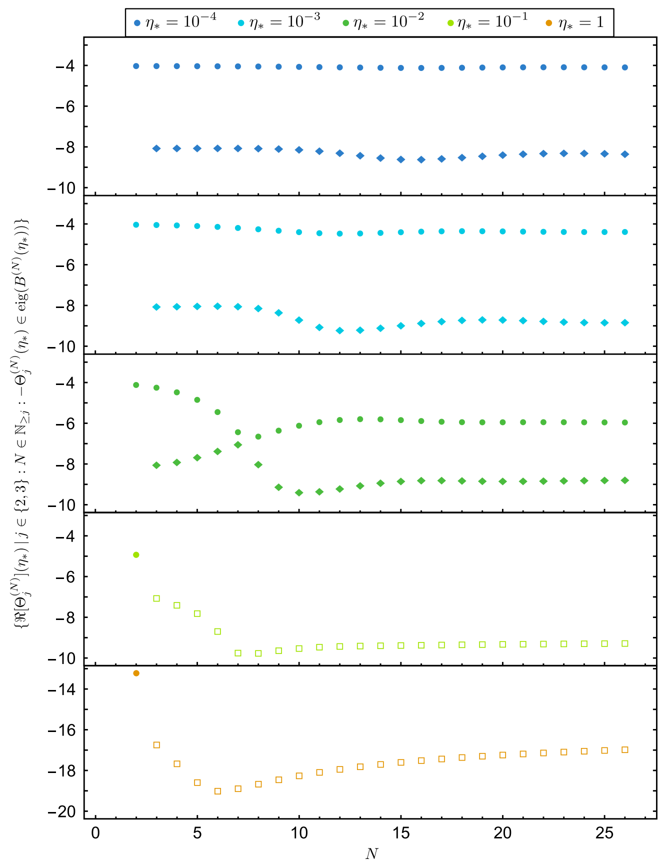

FIG. 8. Real parts of the leading nontrivial critical exponents $\Re[\Theta_2^{(N)}]$ (filled circles) and $\Re[\Theta_3^{(N)}]$ (filled diamonds) corresponding to $u_{2,*}$ and $u_{3,*}$, respectively, as a function of the truncation order N up to $N = 26$ for a logarithmic selection of anomalous dimension values η_* (color code on top of diagram). Complex pairs of critical exponents are marked by open squares. Note the shifted scale of the vertical axis in the lowest plot.

of N. A determination of the spectrum to higher orders thus appears to require a global study of the perturbations. E.g., using an ansatz $w_k(\mathscr{F}) = w_*(\mathscr{F}) + \mathrm{e}^{-\Theta t}\delta w(\mathscr{F})$, a linearization of the flow to leading order in $\delta w(\mathscr{F})$ can give access to the spectrum of the resulting differential operator and thus to all eigenvalues Θ. However, this goes beyond the analytical methods concentrated on in the present work.

In summary: the accessible part of the leading critical exponents covers the trivial exponent [leaving aside the superscript (N)] $\Theta_0 = 4$ for the vacuum energy, as well as the two leading nontrivial ones Θ_2 and Θ_3. The latter are close to their canonical values -4 and -8, respectively, for small $\eta_* > 0$, and are shifted to even more negative values for increasing η_*. Assuming that this pattern holds also for the subleading critical exponents $\Theta_{j\geq 4}$, we find indications that all nontrivial exponents are negative. This implies that all nontrivial perturbations of the fixed point are RG irrelevant, i.e. the fixed function is fully attractive in the long-range limit. The physical implications are discussed below.

VI. EXPLORING THE FULL NONLINEAR SYSTEM AT LEADING-DERIVATIVE ORDER

The flow of the full nonlinear system at leading-derivative order is described by Eq. (20); the corresponding fixed points satisfy the FFE (22). In the preceding sections we have specialized to the reduced system characterized by the two approximations of self duality (A1) and the exclusion of $\mathscr{G}^2$ dependencies (A2). Let us now check the validity of these approximations, by exploring the corrections arising from the inclusion of $\mathscr{G}^2$ contributions to the flow. For this, we now drop the approximation (A2), but keep (A1) in order to exploit the simplicity arising from self-duality for the operator traces. This suffices to include the contributions from the additional operators to the flow and monitor their quantitative relevance.

For a convenient quantitative comparison, we go back to the improper initial conditions and employ the small-field expansion. While the resulting fixed-point candidates presumably are artifacts of the truncation, they allow us to quantify the influence of the $\mathscr{G}^2$-dependent terms. More specifically, we study the following series of increasing truncations:

$$
\begin{aligned}
\mathrm{T1}&: w_k^{(1)}(\mathscr{F},\mathscr{G}^2) = c_k + \mathscr{F} + \frac{1}{2}m_{1,k}\mathscr{F}^2 + \frac{1}{2}m_{2,k}\mathscr{G}^2,\\
\mathrm{T2}&: w_k^{(2)}(\mathscr{F},\mathscr{G}^2) = w_k^{(1)}(\mathscr{F},\mathscr{G}^2) + \frac{1}{2}\sigma_{1,k}\mathscr{F}^3 + \frac{1}{2}\sigma_{2,k}\mathscr{F}\mathscr{G}^2,\\
\mathrm{T3}&: w_k^{(3)}(\mathscr{F},\mathscr{G}^2) = w_k^{(2)}(\mathscr{F},\mathscr{G}^2) + \frac{1}{3}\lambda_{1,k}\mathscr{F}^4 + \frac{1}{3}\lambda_{2,k}\mathscr{F}^2\mathscr{G}^2\\
&\qquad + \frac{1}{3}\lambda_{3,k}\mathscr{G}^4.
\end{aligned}
\tag{53}
$$

In T1 we account for the effects of flow contributions proportional to $\dot{w}_k$, where in T2 we also include nontrivial mixed-derivative inputs proportional to $\dot{w}_k'$. Finally, T3 considers nonvanishing derivatives of w_k to all occurring orders in Eq. (20).

We emphasize that the self-duality approximation (A1) is used only after we have performed all functional derivatives to obtain the Hessian of the action. For instance on the operator level at quartic order in the field strength, we have both terms $\frac{1}{2}m_{1,k}\mathscr{F}^2 + \frac{1}{2}m_{2,k}\mathscr{G}^2$. The evaluation of the final traces using (A1) then corresponds to a projection in coupling space, $m_{1,k}, m_{2,k} \mapsto m_k$, since $\mathscr{F}^2$ and $\mathscr{G}^2$ are identified.

Using the abbreviation $\rho_{q,k}(\eta_k) \coloneqq (2q - \eta_k)$ for $q \in \mathbb{Q}$, the beta functions for truncation T3 read:

$$
\begin{aligned}
\partial_t c_k &\equiv \beta_c = -4c_k + \frac{\rho_{3,k}}{48\pi^2},\\
\partial_t m_k &\equiv \beta_m = -2\rho_{-1,k}m_k + \frac{1}{640\pi^2}(11\rho_{5,k}m_k^2 - 20\rho_{4,k}\sigma_k),\\
\partial_t \sigma_k &\equiv \beta_\sigma = -3\rho_{-\frac{4}{3},k}\sigma_k\\
&\quad - \frac{1}{960\pi^2}(29\rho_{6,k}m_k^3 - 84\rho_{5,k}m_k\sigma_k + 45\rho_{4,k}\lambda_k),\\
\partial_t \lambda_k &\equiv \beta_\lambda = -4\rho_{-\frac{3}{2},k}\lambda_k + \frac{1}{6720\pi^2}(415\rho_{7,k}m_k^4\\
&\quad - 1596\rho_{6,k}m_k^2\sigma_k + 756\rho_{5,k}\sigma_k^2 + 966\rho_{5,k}m_k\lambda_k).
\end{aligned}
\tag{54}
$$

The corresponding beta functions for truncations T2 and T1 can be inferred from (54) by setting $\lambda_k = 0$ for T2 and additionally $\sigma_k = 0$ for T1. Furthermore, the scale-dependent anomalous dimension is given by

$$
\eta_k(m_k) = \frac{10\left(\frac{m_k}{2}\right)}{48\pi^2 + \frac{5}{4}\left(\frac{m_k}{2}\right)}. \tag{55}
$$

At a fixed point, the anomalous dimension is only marginally modified compared to Eq. (30) upon identifying $\frac{m_*}{2}$ with $u_{2,*}$. This identification indeed is consistent: the approximations (A2) and (A1) (performed in this order) reduce the quadratic term of each truncation T1, T2, and T3 to $\frac{m_*}{2}\mathscr{F}^2$ at a fixed point. In the same way, $u_{3,*}$ can be identified with $\frac{\sigma_*}{2}$ and $u_{4,*}$ with $\frac{\lambda_*}{3}$. Interestingly, $\eta_*(m_*)$ is obtained exactly from the expression in (30) by replacing $u_{2,*}$ with $1.2u_{2,*}$.

In Table II, we list all fixed-point values appearing in each truncation. Table III displays the corresponding anomalous dimension and the critical exponents. From this data, we deduce the following results:

(1) As a zeroth-order check, also the full system shows a Gaussian fixed point which is characterized by $\eta_* = 0$ and where all couplings except for the constant c_* vanish identically. The latter takes the same value as in the (A2) and (A1) reduced system. Therefore, the

TABLE II. Fixed-point coordinates for increasing truncations T1, T2, and T3 of theory space of full self-dual nonlinear electrodynamics at leading derivative order using conventional improper initial conditions.

Truncation	c_*	$\frac{m_*}{2}$	$\frac{\sigma_*}{2}$	$\frac{\lambda_*}{3}$
T1	0.0032	0	/	/
	0.0038	−46.108		
T2	0.0032	0	0	/
	0.0035	−25.631	−1787.8	
	0.0041	−66.245	4504.7	
T3	0.0032	0	0	0
	0.0033	−14.657	−1573.9	−107521
	0.0038	−48.181	338.58	249995
	0.0042	−76.839	7955.8	−523227

TABLE III. Anomalous dimension and critical exponents at the fixed points of Table II. The order of the associated fixed points, read from top to bottom, is the same as given in Table II.

Truncation	η_*	Θ_c	Θ_m	Θ_σ	Θ_λ
T1	0	4	−4	/	/
	−1.108	4	4.5096		
T2	0	4	−4	−8	/
	−0.5803	4	4.2625	−5.2259	
	−0.6556	4	4.8249	14.919	
T3	0	4	−4	−8	−12
	−0.3218	4	4.1465	−5.2411	−11.065
	−1.1652	4	4.5389	−5.4326	19.2
	−2.0344	4	5.0627	10.636	33.336

Gaussian fixed function is still and, in fact, exactly, given by $w_{\mathrm{GFP}}(\mathscr{F},\mathscr{G}^2) = w_{\mathrm{GFP}}(\mathscr{F}) = \frac{1}{32\pi^2} + \mathscr{F}$. The critical exponents agree with the canonical mass dimensions of corresponding dimensionful couplings.

(2) In addition to the Gaussian fixed point, we find further non-Gaussian fixed points, the number of which increases from truncation TN to T$(N+1)$ exactly in the same way as in our previous analysis of Sec. IV for the reduced FFE. A direct comparison between the NGFP1 branch of Fig. 1 and the $\frac{m_*}{2}$ column of Table II reveals an initial relative numerical deviation of about 8% while qualitatively showing the same falloff behavior towards the Gaussian fixed point as we increase the truncation order. The same can be seen for the NGFP2 branch and is expected to continue for higher-order non-Gaussian fixed-point classes. Since it is still possible to express each coupling $\sigma_*, \lambda_*, \ldots$ as a function of m_*, small deviations between $\frac{m_*}{2}$ and $u_{2,*}$ imply likewise small deviations between $\frac{\sigma_*}{2}$ and $u_{3,*}$, $\frac{\lambda_*}{3}$ and $u_{4,*}$, etc. The same argument applies to the anomalous dimension.

(3) For the critical exponents, we essentially obtain the same picture as in Sec. IV B. According to Table III, the fixed-point class NGFP**n** exhibits $n+1$ relevant directions. Moreover, Θ_m shows only a minor numerical discrepancy with our previous findings, cf. Fig. 3.

In summary, we found that the additional $\mathscr{G}^2$-dependent terms in the full nonlinear theory to leading-derivative order contribute only quantitatively marginal effects in comparison to the purely $\mathscr{F}$-dependent description of the fixed-point action and the near critical regime. This conclusion holds at least for the small-field expansion using the conventional improper initial conditions. Here, we do not observe any relevant changes in the system's overall behavior and conclude that the assumption (A2) serves a legitimate and efficient approximation in addition to self-duality. It demonstrates the self-consistency of our geometrical line of argument discussed in Sec. III D.

VII. INTERPRETATION SCENARIOS

Let us assume that our nonperturbative results observed for nonlinear electrodynamics also hold beyond the leading-derivative order. Of course, while corrections from higher derivative order are guaranteed to feed back into the lower orders, their contributions to the near critical regime can be expected to remain power-counting suppressed by their higher canonical dimension. This statement is exact at the Gaussian fixed point; and since our solutions for proper initial conditions feature a small anomalous dimension, we expect power-counting arguments to be reliable also in this immediate vicinity of the Gaussian fixed point.

In the following, we discuss several interpretation scenarios. All scenarios are based on the global fixed-point solutions which we found for proper initial conditions, but differ due to additional assumptions or the inclusion of further degrees of freedom.

A. UV-complete nonlinear electrodynamics

The existence of further non-Gaussian fixed points in theory space is a prerequisite for the asymptotic-safety scenario. The global fixed functions which we constructed with proper initial conditions with a positive η_* can then be viewed as scaling solutions of a continuous set of UV fixed points. Each fixed point defines a universality class of UV-complete theories of nonlinear electrodynamics. The long-range behavior in each universality class is then governed by the RG relevant directions of each fixed point.

Interestingly, the only relevant direction is given by the vacuum energy according to our results of Sec. V E, all other directions for which we have reliable data are RG irrelevant. This implies that the nontrivial part of the fixed functional is also IR attractive. For the theory initiated on an RG trajectory emanating from the fixed point in the UV, it remains on the quantum-scale-invariant solution over all scales; in other words, it never leaves the fixed point. The scaling solution, parametrizing the effective action in the form of a nontrivial fixed function, thus governs also the nonlinear interactions of the long-range physics. Apart from the trivial vacuum energy to be fixed by a renormalization condition (and ignored in the following discussion), the full quantum effective action in the low-energy limit is thus given by $\Gamma_{k_0}[A] = \int_{\mathbb{R}^4} w_*(\mathscr{F}, \mathscr{G}^2) \mathrm{d}^4 s$, where s is a dimensionless integration variable and k_0 means some fiducial low-energy scale that serves as a measurement scale for all dimensionful quantities such as the field amplitudes, and $\mathscr{F}, \mathscr{G}^2$ are understood to be measured in units of this scale. Such a theory then has no free parameter and is maximally predictive. In the present scenario, the value of η_* does not play the role of a parameter of the theory, but rather characterizes different theories each forming a universality class labeled by η_*.

In this scenario, it remains a question as to whether this low-energy theory is a genuinely nontrivial theory. For instance, in the reduced version where we have studied only the dependence on the invariant $\mathscr{F}$, the effective Lagrangian $\mathcal{L}(\mathscr{F}) = w_*(\mathscr{F}, 0)$ is a positive and monotonic function in the Euclidean. This suggests that we could perform a nonlinear field transformation of the classical fields, i.e., the expectation values of the quantum fields, $A \to \hat{A}$, such that $\mathcal{L}(\mathscr{F}) = w_*(\mathscr{F}, 0) \equiv \hat{\mathscr{F}}$ assumes the form of the noninteracting Maxwell Lagrangian for the transformed gauge field $\hat{A}$. However, since our results of Sec. VI suggest that the scaling solution also depends nontrivially on the invariant $\mathscr{G}$, a transformation $w_*(\mathscr{F}, \mathscr{G}^2) \to \hat{\mathscr{F}}$ does most likely not exist. In this case, the scaling solution does represent a nontrivial interacting theory on macroscopic scales.

It is instructive to study the effective action also in Minkowski space. Using k_0 as an IR reference scale, the effective Lagrangian expressed in terms of the dimensionless Minkowski-valued invariants reads to lowest order:

$$\frac{\bar{\mathcal{L}}(\bar{\mathcal{F}}, \bar{\mathcal{G}}^2)}{k_0^4} = -\mathcal{F} - \frac{1}{2} m_1 \mathcal{F}^2 + \frac{1}{2} m_2 \mathcal{G}^2. \tag{56}$$

This form of the Lagrangian is well known from the weak-field analysis of the Heisenberg-Euler action [5]. The leading-order nonlinear coefficients $m_{1,2}$ can be related to the properties of light propagation in an external field, cf. [9,126,127]. Now, causality can be argued to impose constraints on the values of these coefficients [128,129]; more precisely, requiring that the phase velocities of low-energy photons do not exceed the vacuum speed of light, implies [9,128,129]

$$m_2 - m_1 \geq 0. \tag{57}$$

While the more relevant quantity for causality actually is the front velocity which is given by the high-frequency limit of the phase velocity, it has been argued that the front velocity is always bound from below by the low-frequency phase velocity for a nonamplifying ground state [129–131]. Therefore, our resulting effective action needs to satisfy the causality constraint Eq. (57). For the present truncation, we find $m_2 - m_1 = 0$ as a result of the self-dual approximation rather than of a full calculation. Whether or not the fixed-point action does satisfy all necessary causality constraints hence requires further investigation going beyond the self-dual techniques used in the present work.

In summary, we conclude that our fixed functions constitute a UV-complete version of nonlinear electrodynamics which is essentially parameter free, as the long-range physics is also governed by the scaling solution. We emphasize that this scenario does not solve the triviality/ Landau-pole problem of QED, since the latter arises from interactions with matter which are not included in the present scenario.

B. Low-energy QED effective action

Within the context of QED, the inclusion of electron matter degrees of freedom modifies the picture for pure nonlinear electrodynamics in several ways. First, electron fluctuations induce an anomalous dimension for the gauge field. In QED, we have at one-loop order $\eta = \frac{2}{3\pi}\alpha$. For $\alpha \approx 1/137$, this yields $\eta \approx 0.00155$. This value is well within the regime where we have been able to construct global fixed functions.

A second modification arises from the fact that the electronic fluctuations, of course, also contribute to the flow of the effective action. On the level of the flow equation, this contribution serves as an inhomogeneous source term, depending on the gauge field, but not on the field-strength potential w_k. For instance, integrating only this source contribution leads to the one-loop Heisenberg-Euler effective action [5]. From a perturbative viewpoint, the contributions arising from integrating the flow induced by the terms depending nonlinearly on w_k and its derivatives correspond to resumming higher-loop contributions. In the perturbative domain, these terms are subleading compared to the one-loop terms at least in the small-field domain.

The situation is less clear at large field amplitudes where the size of the amplitude can make up for a small-coupling value. If $\eta \approx$ const for a sizable number of scales, the IR attractive nature of our fixed-point solution can win out over the matter induced direct terms. In this case, we expect the Minkowskian-valued effective Lagrangian at some low-energy scale k_0 to assume the asymptotic strong-field form, cf. Eq. (37),

$$\frac{\bar{\mathcal{L}}(\bar{\mathcal{F}}, \ldots)}{k_0^4} \sim -\mathcal{F}^{\Delta(\eta)} \quad (\bar{\mathcal{F}} \to \infty), \quad \Delta(\eta) = \frac{4}{4+\eta}. \tag{58}$$

Using $\eta = \frac{2}{3\pi}\alpha$ and expanding Eq. (58) in powers of α for the example of a magnetic background B, we obtain to order α at large fields:

$$\frac{\bar{\mathcal{L}}(B)}{k_0^4}\bigg|_\alpha \sim \frac{\alpha}{6\pi}\mathsf{B}^2\ln(\mathsf{B}). \tag{59}$$

Identifying k_0 with the electron mass scale, $k_0 = m_e$, this result corresponds precisely to the strong magnetic field limit of the one-loop Heisenberg-Euler Lagrangian. Incidentally, if we had used the correct two-loop anomalous dimension, we would have found the correct two-loop strong-field limit in line with an argument relating the strong-field limit with the trace anomaly of the energy-momentum tensor [132,133].

While we consider this observation as reassuring for our result for the strong-field limit, note that the argument based on the trace anomaly relies on identifying the RG scale with the strong-field scale. This identification is also used for leading-log resummations of the perturbative strong-field series [37,38]. In complete analogy to the leading-log resummation for the running coupling, such resummations lead to a Landau-pole singularity in the effective action at exponentially large field strength $\bar{\mathsf{B}} \sim m_e^2 e^{3\pi/\alpha}$.

By contrast, the resummation implicitly performed by the functional RG flow equation does not rely on scale identification. Thus, there is no reason for the Landau pole of the running coupling at high momenta to be translated into a similar singularity for large field strength. In fact, the asymptotic form suggested by Eq. (58) as well as our full global solutions are free of any singularity. Therefore, we conjecture that Eq. (58) describes the strong magnetic field limit of the 1PI effective action of QED. We emphasize that this is not in contradiction with strong-field results for the Heisenberg-Euler action based on the Schwinger functional dominated by one-particle reducible (1PR) diagrams [40,41].

More precisely, we consider Eq. (58) to hold as long as the magnetic field is the dominating scale with all other energy scales (test particles, photons, etc.) being much smaller and in the perturbative domain. Also, Eq. (58) does not hold for the equivalent electric case which is dominated by Schwinger pair production and an energy transfer to particle degrees of freedom. We emphasize that the singularity-free strong-field limit Eq. (58) does also not solve the Landau pole problem nor render QED UV complete, since the high-energy behavior remains dominated by η growing large (with a Landau-pole singularity as a perturbative artifact) and thus a UV limit being precluded by a causally disconnected chirally broken phase [19,22].

As a last comment, it is tempting to speculate if our fixed functional may have any relevance for the deep IR limit of QED far below the electron mass threshold, since the matter induced photon self-interactions render the effective theory an interacting one. However, at the same time, the leading matter dominated contributions to the anomalous dimension decouple below the electron mass threshold, $\eta|_{k\ll m_e} \to 0$, such that the mechanisms leading to our nontrivial solutions disappear towards the deep IR. Of course, a more precise answer requires a full (numerical) solution of the RG flow for w_k including the matter source terms and the electron decoupling.

C. High-energy hypercharge sector of the standard model

In the full standard model, our results may find application for the hypercharge sector above the electroweak scale. Here, the anomalous dimension at one-loop order is given by $\eta_Y = \frac{41}{3}\frac{g_Y^2}{(4\pi)^2}$, cf. [134]. At the electroweak scale, we have $g_Y \approx 0.36$ for a NNLO fixing at the top mass scale [135] increasing mildly to $g_Y \approx 0.48$ at about the Planck scale. This corresponds to values of the anomalous dimension of about $\eta_Y \approx 0.01\ldots0.02$ and thus well in the range of values for which we find global solutions with a strong-field asymptotic limit given by Eq. (58) in the magnetic field direction.

Our conclusion for this application is therefore similar to that for QED discussed in the preceding subsection: provided the global fixed function is sufficiently attractive under the RG flow, the 1PI effective action exists globally for any value of the hypercharge magnetic field.

VIII. CONCLUSIONS

We have investigated the renormalization flow in the theory space of Abelian quantum gauge fields, i.e., nonlinear electrodynamics, using the nonperturbative functional RG. For this, we have concentrated on the RG flow of the 1PI effective action to leading order in a derivative expansion. The resulting flow equation (20) for the effective Euclidean Lagrangian or field-strength potential represents a main result of our work and generalizes a previous result for the magnetic theory space [63] to the full space of nonderivative invariants $\mathscr{F}$ and $\mathscr{G}^2$.

In order to explore the phase diagram in this theory space on nonlinear electrodynamics, we focus on the possible existence of fixed points and construction of corresponding fixed functionals. Most of our explicit results refer to an approximation scheme relying on simplifications for a self-dual choice of the electromagnetic field; in addition, we provide evidence that our results are only mildly modified if these approximations are dropped. For the construction of fixed-point actions, we have used two approaches: The first one corresponds to a conventionally used small-field expansion. This procedure corresponds to an implicit use of improper initial conditions for the fixed-point equation yielding a large number of artifact fixed points. We observe a pattern different from Wilson-Fisher-like systems but similar to that discovered for shift-symmetric theories [63,64]. Following [64], we arrive at the conclusion that none of the nontrivial fixed points using improper initial conditions approximates a valid fixed point in full theory space.

In the second approach, we have used proper initial conditions and studied small- and large-field expansions. Emphasizing the importance of the criterion of global existence, i.e., the absence of singularities of the effective action in amplitude space, we have been able to construct full fixed functionals. In fact, our construction provides evidence for the existence of a continuous family of fixed points as a function of the anomalous dimension $\eta_* > 0$ of the field amplitude. This quantity parametrizes naturally the initial condition of the fixed-function equation at small field amplitude as well as governs the large-field asymptotics. We have also checked whether a quantization of the fixed-function solutions is induced by the presence of a movable singularity in the fixed-function equation. Our results are compatible with the absence of a movable singularity thus facilitating the presence of a continuous solution family. Still our results suggest the existence of a critical value of η_* above which no valid solution exists.

Finally, we have explored the critical region of the fixed-point solutions. Our results are compatible with this continuum of fixed points being fully IR-attractive apart from the trivial vacuum energy. However, a more definite conclusion requires also a global analysis of the perturbations around the fixed point beyond our analysis so far.

We have discussed several scenarios for which our results could be relevant. Imminently, these non-Gaussian fixed points can serve to define an interacting theory of nonlinear electrodynamics without matter with the fixed-point action being equivalent to the long-range action as the fixed point is fully IR attractive. We also offer several applications to scenarios that include matter as the source of a nontrivial anomalous dimension. In this cases, our fixed-point action has the potential to dominate the magnetic strong-field limit of the effective action. We consider this an attractive feature as it removes the puzzle of a Landau-pole type singularity in the leading-log resummed strong-field limit based on scale identification. As our flow equation approach does not need scale-identification arguments and goes beyond the leading-log resummation, we consider the existence of a global Lagrangian as being in line with the fact that magnetic fields do not transfer energy to charged fluctuations; therefore the strong magnetic field limit can behave differently from the high-momentum limit of the theory.

We emphasize that our observation of fixed points in pure nonlinear electrodynamics does not resolve the triviality problem of QED. The latter is tightly linked to charged particle fluctuations not being part of our pure Abelian gauge theory setting. Also, we do not observe an immediate mechanism that could balance the charged fluctuations within nonlinear electrodynamics. Still, the family of non-Gaussian fixed points observed in this work could play a useful role in models with gauge-kinetic or nonminimal interactions to other particle sectors, potentially contributing to mechanisms of UV completion.

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