ANALYTICAL AND NUMERICAL STUDIES ON KICKED BEAMS IN THE CONTEXT OF HALF-INTEGER STUDIES*

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Abstract

In the context of half-integer studies an investigation of the kicked beam dynamics has revealed surprising characteristics. The coupling of space charge with chromaticity in addition to usual damping/non-damping dynamics, exhibits new properties typical of a linear coupling. This article gives the status of these studies which were carried out with analytical and numerical approaches as well as preliminary results of experimental investigations in the CERN PS Booster.

LANDSCAPE

The delivery of high-intensity beams is constrained by several effects, among which the effect of machine error resonances is of significant impact [1]. Operational scenarios at injection should prevent the space charge tune-spread from overlapping with any machine resonance and, as the lower-order resonances are the strongest, they should be avoided or corrected.

Gradient errors affect single particle dynamics in such a way that they alter optical functions and generate beta-beating [2]. They also create an instability stop-band, where the particle's motion is subject to an exponential amplitude growth. This view considers the unperturbed lattice joint with the distribution of linear errors as "another linear lattice", which can be treated with the usual optics tools for stable dynamics [3], and also with ad-hoc optics for unstable single-particle dynamics [4].

For a high-intensity beam, dynamics are more complex. Frank Sacherer has shown that for a KV coasting beam, the resonance location is shifted with respect to the one of the single particle resonance [5]. This effect is "coherent" as dynamics are completely determined by global quantities and not by a single particle. On the other hand, this result is fully valid for a perturbative gradient error and for a KV distribution. The beam response is not localized for non-perturbative errors and non-KV beam distributions, and a typical spread of the beam response results from the resonance dynamics. The disentangling of the effect created by a half-integer resonance is of interest to several laboratories such as GSI [6–8], CERN [9, 10], RAL [11], and FNAL [12, 13].

Recently, an experimental campaign has been carried out at the CERN PSB on the effect of half-integer resonances on an intense coasting beam [14, 15]. Several scenarios have been investigated, from the effect for fixed accelerator parameters to the investigation of a dynamical crossing of the half-integer resonance. The associated simulation effort to explain and interpret the experimental findings has revealed an anomalous pattern, which brought the attention to the dynamics of an intense coasting beam performing coherent oscillation during the resonance crossing, and on the effect of space charge on coherent/decoherent dynamics. Studies on the instability of coasting beams are presented in Refs. [16, 17], and a discussion on its relation to the half-integer resonance is presented in Ref. [18]. Studies of decoherence in bunched beams have also been carried out in Refs. [19, 20].

We focus here on the dynamics of a kicked coasting beam when subject to pure direct space charge fields, i.e. we neglect the effect of impedance and other collective effects.

EFFECT OF THE CHROMATICITY

A displaced coasting beam in the phase space, when subject only to chromatic effects, can be analyzed from the single particle dynamics point of view. Each beam particle performs a specific harmonic oscillation, which has a main frequency plus a shift proportional to the particle momentum offset. The small focusing shift does not affect the optics, as we carry out this investigation off the half-integer resonance (and off-momentum beta-beating is neglected). The resulting effect on the center of mass is that the amplitude of oscillation is modulated by the function $\Lambda(\tau s)$, where

$$\tau = \left(\frac{Q_{x0} \xi_x \sigma_p}{R} \right). \tag{1}$$

Here, Q_{x0} is the tune, ξ_x is the normalized chromaticity, R the accelerator radius, and σ_p the rms momentum spread. The function $\Lambda(u)$ is given by

$$\Lambda(u) = \int \cos(u\lambda) g(\lambda) d\lambda \tag{2}$$

and the function $g(\lambda)$ is the normalized distribution of the momentum offsets satisfying the properties

$$g(-x) = g(x), \qquad \int g(x) dx = 1,$$
$$\int x g(x) dx = 0, \qquad \int x^2 g(x) dx = 1.$$

For a uniform distribution of $(\delta p/p)$ we have

$$\Lambda(u) = \frac{\sin\left(\sqrt{3} u\right)}{\sqrt{3} u}, \tag{3}$$
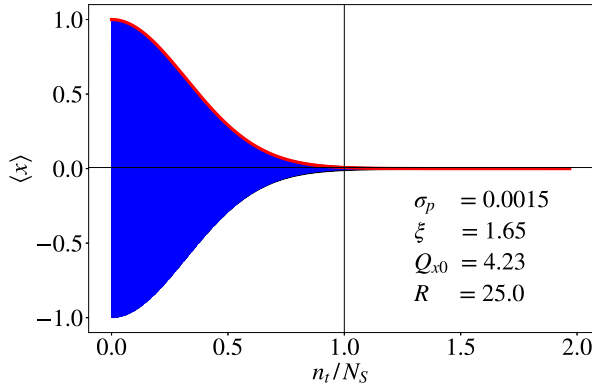

Figure 1: Example of oscillation decoherence of $\langle x \rangle$ as a function of n_t/N_S (blue curve). At turn $n_t = N_S$, the damping reaches 0.72%. The function $\Lambda(\pi\, n_t/N_S)$ is shown in red.

while for a Gaussian we find

$$\Lambda(u) = \exp\left(-u^2/2\right). \tag{4}$$

Note that for the uniform case $\Lambda(u)$ exhibits damped oscillations of periodicity $\Delta u = \pi/\sqrt{3}$; the amplitude of which is also dampened by $1/(\sqrt{3}u)$. In the Gaussian case, there is no periodicity of Λ, but only damping: for $u = 1$ we have $\Lambda(1) = 0.6$, and for $u = 3$ we have $\Lambda(3) \simeq 0.011$.

We call for convenience

$$S = \pi \frac{R}{Q_{x0}\xi_x\sigma_p} \tag{5}$$

so that the damping of the center of mass due to chromaticity reads $\Lambda(\pi s/S)$. We also define the rms chromatic tuneshift as

$$\delta Q_{x,\xi} = Q_{x0}\xi_x\sigma_p, \tag{6}$$

to which we associate the number of turns

$$N_S = \frac{1}{2\,\delta Q_\xi}. \tag{7}$$

The rate of the center of mass damping is characterized by the transport length in which the rms chromatic tune-shift creates a transverse phase advance shift of π. This length is equal to S, corresponding to N_S turns. For a Gaussian distribution of $(\delta p/p)$, after N_S turns, the center of mass oscillations are fully damped (see Fig. 1). For a unifom distribution, the beam is subject to de-coherence and re-coherence progressively damping as $\propto (s/S)^{-1}$. We keep $N_S = \frac{1}{2\,\delta Q_\xi}$ as reference number of turns to characterize the the timescale of the processes in the following discussion.

TWO-PARTICLE MODEL

We first discuss a simplified model to understand possible relevant mechanisms. We consider two particles of a coasting beam, traveling in a constant focusing lattice, and discuss dynamics in the horizontal plane only. These

particles, labelled 1 and 2, have momentum offsets $(\delta p/p)_{1,2}$ obeying:

$$\left(\frac{\delta p}{p}\right)_1 = -\left(\frac{\delta p}{p}\right)_2. \tag{8}$$

Each of those will have a betatron oscillation frequency shifted by

$$\Delta Q_{x,1/2} = \xi\left(\frac{\delta p}{p}\right)_{1/2}, \tag{9}$$

and so

$$\delta Q_\xi = \left|\Delta Q_{x,1/2}\right|. \tag{10}$$

Next, we apply a force to them that depends on their reciprocal distance. This force is modeled to resemble the actual space charge force in a coasting beam, but with the property that it is zero in the center of mass of these two particles (which is taken as the center of mass of the coasting beam). The form of the force exerted on particle 1 is

$$F_{12} = \frac{x_1 - x_2}{|x_1 - x_2|} f(x_{12}), \tag{11}$$

with

$$f(x_{12}) = \lambda \frac{|x_1 - x_2|}{d^2 + (x_1 - x_2)^2}. \tag{12}$$

Inverting indices 1 and 2 in Eqs. (11) and (12), we find correspondingly the force on particle 2. Here d is a characteristic distance, and λ is a dimensionless parameter controlling the magnitude of the force. This force satisfies the third law of dynamics, and has the property that

$$\frac{|x_1 - x_2|}{d} \ll 1 \;\Rightarrow\; f(x_{12}) \simeq \frac{\lambda}{d^2}|x_1 - x_2|. \tag{13}$$

Instead, we have

$$\frac{|x_1 - x_2|}{d} \gg 1 \;\Rightarrow\; f(x_{12}) \simeq \lambda\frac{1}{|x_1 - x_2|}. \tag{14}$$

By re-scaling the coordinate with $z = (2/d)\,x$, defining the center of mass $z_{cm} = (z_1 + z_2)/2$, and the location of one particle $z = z_1 - z_{cm}$ (we drop the particle index) we obtain the following equations of motion

$$\begin{aligned}\ddot{z}_{cm} + kz_{cm} + \Delta k z &= 0,\\ \ddot{z} + kz + \Delta k z_{cm} &= 2\frac{\lambda}{d^2}\frac{z}{1 + z^2}.\end{aligned} \tag{15}$$

In these equations, $k = Q_{x,0}^2/R^2$ is the focusing strength of the transport structure, $\Delta k \simeq 2k\delta Q_\xi/Q_{x,0}$, is the perturbation due to the chromaticity from the $\pm\delta p/p$ assigned to the two particles. We observe that Eqs. (15) are coupled by a linear coupling of harmonic zero with strength Δk. The term with λ represents the strength of the space charge.

No Space Charge: Full Linear Coupling Dynamics

If the space charge force is small, i.e. if $\lambda \simeq 0$ then, in good approximation

$$\begin{aligned}\ddot{z}_{cm} + kz_{cm} + \Delta kz &= 0, \\ \ddot{z} + kz + \Delta kz_{cm} &= 0.\end{aligned} \quad (16)$$

We consider an initial condition of a kicked beam such that $z_{cm} \gg z$. Therefore we are in a classic condition for an "emittance exchange" between the emittance a of the dynamical variable z, and the emittance a_{cm} of the dynamical variable z_{cm}. As we have only two particles, the amplitude of the center of mass exhibits periodic decoherence and re-coherence, which here is the process of "emittance exchange". From the theory of the linear coupling, but also from the equation of motion of each particle, we can obtain the main features of this process, which are:

1. the system is always resonant,

2. $a + a_{cm}$ is preserved,

3. the periodicity of the exchange N_{exch} is given by Δk, which yields $2\,\delta Q_\xi N_{exch} = 1$, that is $N_{exch} = N_S$.

The meaning of this exchange, if translated to a full beam, is that the coherent oscillations of the center of mass would damp to zero.

Partial Linear Coupling Dynamics

Taking the linear part of the space charge force to the left-hand side, Eqs. 15 can be rewritten as

$$\begin{aligned}\ddot{z}_{cm} + kz_{cm} + \Delta kz &= 0, \\ \ddot{z} + k_d z + \Delta kz_{cm} &= -2\frac{\lambda}{d^2}\frac{z^3}{1+z^2},\end{aligned} \quad (17)$$

Hence, if the space charge force is small but not too small, we can neglect the nonlinear term on the right-hand side. In Eqs. 17, the depressed focusing strength on z is

$$k_d = k - 2\lambda/d^2. \quad (18)$$

In this context, k creates the tune $Q_{z,cm}$, and k_d creates the tune Q_z. The term $-2\,(\lambda/d^2)z$, implicit in Eqs. (17), creates the incoherent linear space charge force responsible for the incoherent tune-shift

$$\Delta Q_{z,cm} = \Delta Q_{x,sc} \simeq -\lambda R^2/(Q_{x,0}\,d^2), \quad (19)$$

which detunes this system from the linear coupling resonance: in fact, the distance from the resonance is $Q_{z,cm} - Q_z = -\Delta Q_{x,sc}$. Therefore, in a first approximation, the full linear coupling dynamics is prevented by the space charge that "detunes" the emittance exchange process. From the theory of linear coupling, (see [21] for a review), we can use the formulas that yield the amount of emittance exchange and its periodicity N_{exch} when the tunes are not exactly on the resonance. If the distance from the resonance is not too
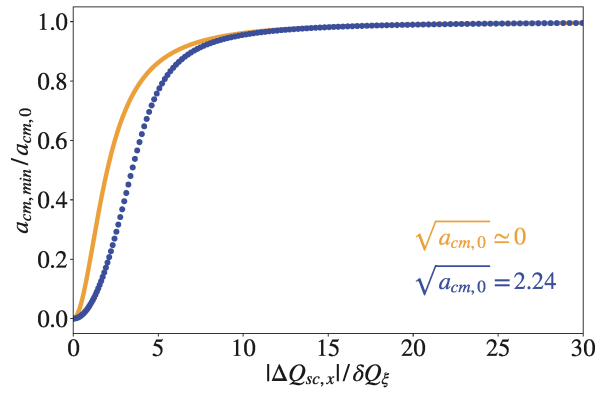

Figure 2: Miminum a_{cm} (blue) as a function of $|\Delta Q_{sc}|/\delta Q_\xi$. In orange is the small kick limit.

large, the emittance exchange reads as follows, in terms of the quantities in Eq. (17)

$$\frac{a_{max}}{a_{cm,0}} = \left[1 + \left(\frac{k_-}{\Delta k}\right)^2\right]^{-1} \quad (20)$$

where $a_{cm,0}$ is the initial "emittance" of the center of mass, and a_{max} the maximum emittance of z during the exchange process. Here,

$$k_- = \frac{k - k_d}{2} = -k\frac{\Delta Q_{x,sc}}{Q_{x,0}}. \quad (21)$$

The periodicity of the emittance exchange N_{exch} is instead given by

$$\sqrt{1 + \left(\frac{k_-}{\Delta k}\right)^2}\frac{N_{exch}}{N_S} = 1. \quad (22)$$

The quantity

$$\left|\frac{k_-}{\Delta k}\right| = \frac{1}{2}\frac{|\Delta Q_{sc}|}{\delta Q_\xi} \quad (23)$$

determines both the amount of exchange and its wavelength when scaled with N_S.

Perturbative Analysis

Perturbative analysis allows us to retrieve general properties of Eqs. (15). In particular, we can find the dynamics of the "slowly varying constants" defined for this problem as a and a_{cm}, keeping the quantity $a + a_{cm}$ constant in the most general case. In Fig. 2 we show the minimum a_{cm} as predicted by the perturbative theory. The orange curve is the "small kick limit" where $a_{cm,0} \to 0$. In blue the case for $\sqrt{a_{cm,0}} = 2.24$, which is the case for an initial beam oscillation of amplitude $x_{ini,max}/d = 2.24$.

Summary and Intriguing Prospects

The analysis of this model allows us to draw the following conclusions:

1. The dynamical variables z, z_{cm} are linearly coupled through the chromaticity, and the full emittance exchange is associated with the beam center of mass decoherence.
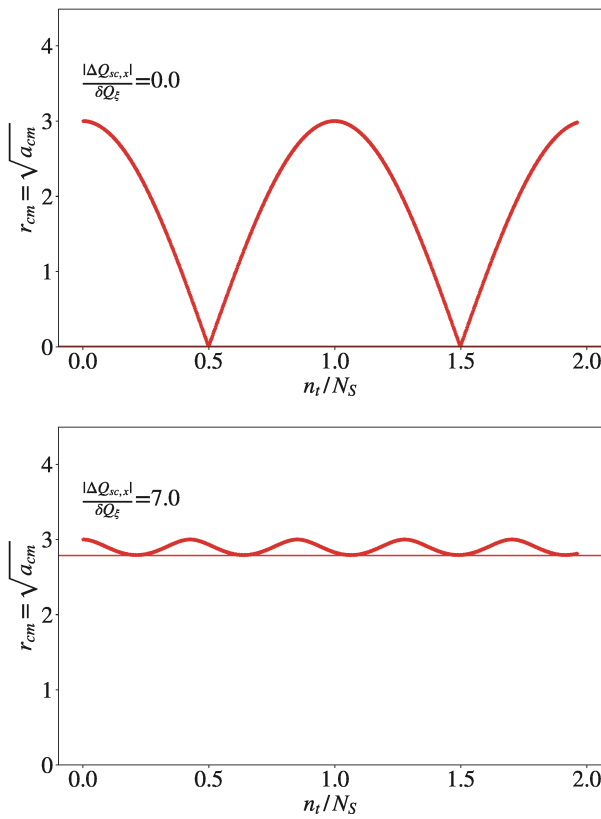
Figure 3: Top: here the space charge is absent and full emittance exchange take place. The quantity $\sqrt{a_{cm}}$ oscillate between maximum and zero with periodicity equal to $N_{exch} = N_S$. Bottom: Now $|\Delta Q_{sc}|/\delta Q_\xi = 7$ and the exchange is strongly damped. However, a residual oscillation is still visible: this has been experimentally observed.

2. In absence of space charge the periodicity of the exchange is of $N_{exch} = N_S$ turns;

3. In the presence of space charge, its linear part de-tunes the system off the linear coupling, and the formulas for its exchange and wavelength N_{exch}/N_S are functions of $|\Delta Q_{sc}|/\delta Q_\xi$;

4. The perturbative analysis allows us to conclude that $a + a_{cm}$ is constant also when the nonlinear term of Eq. (15) is fully accounted for.

This study has an intriguing aspect as the emittance exchange process also allows for a partial exchange, which reflects on the time evolution properties of a, a_{cm}. This means that between the *full beam damping* regime and the one of *full space charge compensation* of the damping, there should exist a regime in which the two dynamical variables "beat", and this makes this effect measurable. We show this situation in Fig. 3.

FULL BEAM MODEL

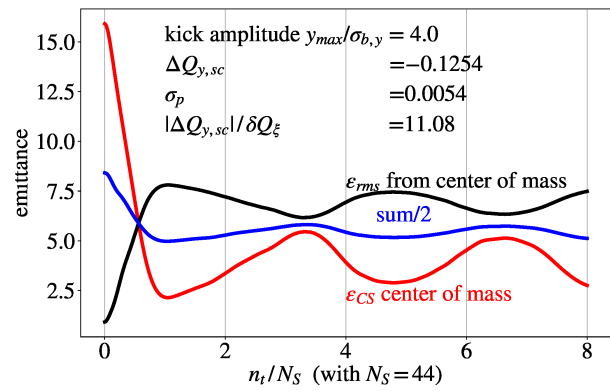The same approach can be employed for analyzing the full beam, and a system analog to Eqs. (15) can be worked out. For small amplitude kicks we obtain again two linearly coupled differential equations which will express the dynamics similarly to Eqs. (15), with the difference that now the quantities that will be exchanged are the center of mass $\langle x \rangle$, and the quantity $\langle x \delta p/p \rangle$. Although *inconsistent*, the resulting equations of motion for small nonlinear terms are similar to Eq. (16) with the same coupling and focusing strength. Hence, the same scaling as for the two-particle model can be retrieved.

Figure 4: Example of effect of the coupling between "coherent" motion of the center of mass, ϵ_{CS} (red) and the "incoherent" motion of the beam with respect to the center of mass ϵ_{rms} (black). The coupling is quite evident.

PIC SIMULATIONS

Further investigations employing Particle-In-cell simulations, without wakefield or impedance, have been carried out to verify the analytical model scaling prediction. We use a constant focusing model, which has the following feature: $R = 25$ m, $Q_{y,0} = 4.23$, $\xi_y = 1/2$ (the natural chromaticity for a constant focusing transport channel). We consider a beam with a Gaussian distribution of $\delta p/p$, and rms emittance $\epsilon_x = 1$ mm mrad. In Fig. 4 we show one example of the center of mass beating dynamics for a kicked beam with $\sqrt{a_{cm,0}} = 4$ (red curve). In black the rms beam emittance from the center of mass. The coupling between incoherent (black) and coherent (red) is quite evident. In Fig. 5 we show a systematic study for several space charge tune-shifts, but for an initial kick in the "small kick limit", as of $y_{max}/\sigma_{b,y} = 0.2$. We see that all simulation results overlap in a curve that is well defined, so confirming the scaling from the analytic model.

FIRST EXPERIMENTAL RESULTS

Inspired by these intriguing predictions a dedicated campaign has been carried out at the CERN PSB, with the purpose of finding experimental evidences of the coupling between coherent and incoherent dynamics. We investigated the evolution of a vertically kicked coasting beam. The observable we investigated is the beam center of mass, measured with the BBQ system [22]. In these
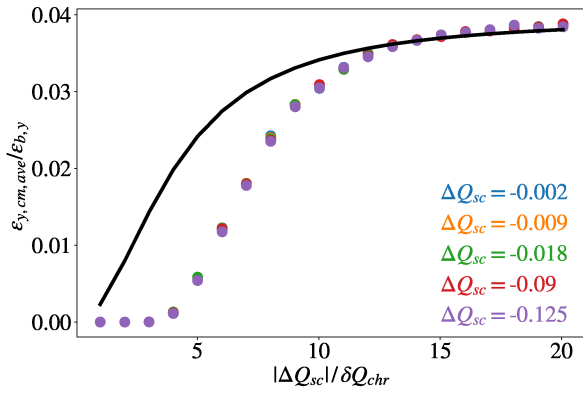
Figure 5: Average emittance of the center of mass for several space charge tune-shifts and chromaticities. Note that all the results overlap. In all these simulations the initial kick yields $y_{max}/\sigma_{b,y} = 0.2$. Here $\sigma_{b,y}$ is the initial rms beam size. The black curve shows the minimum a_{cm} predicted by the two-particle model.
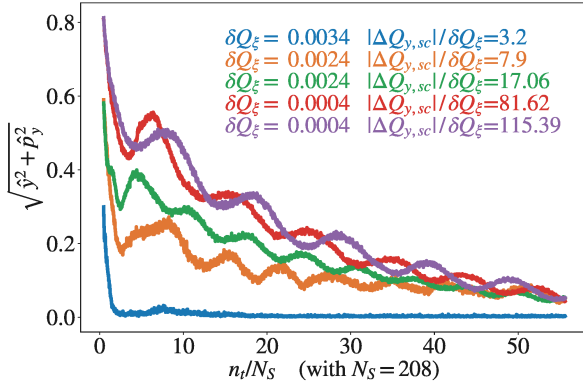

Figure 6: Example of measurement results. The center of mass beating is distinctly visible. For very small $|\Delta Q_{y,sc}|/\delta Q_\xi$ there is full damping. For larger $|\Delta Q_{y,sc}|/\delta Q_\xi$ the damping is reduced.

measurements, the RF system was *not activated*. Turn-by-turn measurements allowed us to find $\tilde{y}_{n_t}$, which is proportional to the real center of mass amplitude y_{n_t}. Hence, we retrieved the scaled phase space coordinate $\tilde{y}_{cm}, \tilde{y}'_{cm}$ from the analysis of pairs $(\tilde{y}_{cm,n_t}, \tilde{y}_{cm,n_t+1})$, and applying a Courant-Snyder transformation we obtain the scaled normalized coordinates $\hat{y}, \hat{p}_y$. Then we compute the radius $\sqrt{\hat{y}^2 + \hat{p}_y^2} \propto \sqrt{a_{cm}}$. In Fig. 6 we show the evolution of a few kicked beams, all with the same maximum reading from the BPM system, having different intensities and momentum spread. The beating of the center of mass is observed in all but the blue trace (strong damping).

STATUS AND OUTLOOK

We have found experimental indications of a periodic energy exchange between coherent and incoherent dynamics.

More effort is necessary to analyze the experimental data into a coherent picture with simulations and theory.

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