ISBN: 978-3-95450-253-0 ISSN: 2673-5571 doi:10.18429/JACoW-HB2023-THA2C3

Content from this work may be used under the terms of the CC-BY-4.0 licence (© 2023). Any distribution of this work must maintain attribution to the author(s), title of the work, publisher, and DOI

PERIODIC SOLUTION FOR TRANSPORT OF INTENSE AND COUPLED COASTING BEAMS THROUGH QUADRUPOLE CHANNELS

C. Xiao*, L. Groening,
GSI Helmholtzzentrum für Schwerionenforschung GmbH, Darmstadt, Germany

Abstract

Imposing defined spinning to a particle beam increases its stability against perturbations from space charge. In order to fully explore this potential, proper matching of intense coupled beams along regular lattices is mandatory. Herein, a novel procedure assuring matched transport is described and benchmarked through simulations. The concept of matched transport along periodic lattices has been extended from uncoupled beams to those with considerable coupling between the two transverse degrees of freedom. For coupled beams, matching means extension of cell-to-cell periodicity from just transverse envelopes to the coupled beam moments and to quantities being derived from these.

INTRODUCTION

Preservation of beam quality is of major concern for acceleration and transport especially of intense hadron beams. This aim is reached at best through provision of smooth and periodic beam envelopes, being so-called matched to the periodicity of the external focusing lattice. The latter is usually composed of a regular arrangement from solenoids or quadrupoles. For the time being, the quality of matching has been evaluated through the periodicity of spatial beam envelopes. This is fully sufficient as long as there is no coupling between the phase space planes (for brevity "planes"), neither in beam properties nor in lattice properties.

Spinning of beams is a very promising tool to further augment accelerator performance. It requires coupling between planes and thus imposes dedicated efforts for proper matching to periodic lattices. Beam matching with coupling between the horizontal and longitudinal planes has been investigated in [1]. Special cases of beams with zero four-dimensional emittances have been treated in [2] The present work is on the development and demonstration of a method to assure rms-matched transport of intense beams with considerable transverse coupling, an issue being addressed conceptually in [3]. It partially implements the early concept, i.e. tracking of moments, into a procedure to obtain full cell-to-cell four-dimensional (4D)-periodicity. Through simulations it is shown that the lattice periodicity is not just matched by the two transverse envelopes but also by the beam rms-moments that quantify coupling. To this end, an iterative procedure towards the periodic solution is applied. It starts from determining the solution with zero current, using a method that is applied later also to beams with current.

Coupled beams inhabit ten independent second-order rms-moments. They are summarized within the symmetric beam moments matrix

$$C := \begin{bmatrix} \langle xx\rangle & \langle xx'\rangle & \langle xy\rangle & \langle xy'\rangle \\ \langle x'x\rangle & \langle x'x'\rangle & \langle x'y\rangle & \langle x'y'\rangle \\ \langle yx\rangle & \langle yx'\rangle & \langle yy\rangle & \langle yy'\rangle \\ \langle y'x\rangle & \langle y'x'\rangle & \langle y'y\rangle & \langle y'y'\rangle \end{bmatrix}. \tag{1}$$

Four of its elements quantify beam coupling. Beams are x-y coupled if at least one of these elements is different from zero.

A simple way to impose spinning to a beam is to pass it through an effective half solenoid. Although half solenoids do not exist due to $\vec{\nabla}\cdot\vec{B} = 0$, their effect can be imposed by particle creation inside the solenoid or by changing the beam charge state inside the solenoid [4,5] and demonstrated experimentally in [6].

The first part of the transport matrix S^h of an effective half solenoid is given by the matrix $S_\rightarrow$ of the main body of the solenoid of effective length L, comprising just the pure longitudinal magnetic field B_s

$$S_\rightarrow = \begin{bmatrix} 1 & \frac{\sin(2KL)}{2K} & 0 & \frac{1-\cos(2KL)}{2K} \\ 0 & \cos(2KL) & 0 & \sin(2KL) \\ 0 & -\frac{1-\cos(2L)}{2K} & 1 & \frac{\sin(2KL)}{2K} \\ 0 & -\sin(2KL) & 0 & \cos(2KL) \end{bmatrix}, \tag{2}$$

with $K := B_s/[2(B\rho)]$ (Larmor wave number) and $(B\rho)$ as beam rigidity.

The second part of S^h is from the fringe field matrix $S_\downarrow$ of the solenoid exit

$$S_\downarrow = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -K & 0 \\ 0 & 0 & 1 & 0 \\ K & 0 & 0 & 1 \end{bmatrix}, \tag{3}$$

and the total matrix of the half solenoid is the product of both matrices

$$S^h = S_\downarrow \cdot S_\rightarrow = \begin{bmatrix} S^h_{xx} & S^h_{xy} \\ S^h_{yx} & S^h_{yy} \end{bmatrix}. \tag{4}$$

The determinants of the diagonal sub-matrices S^h_{xx} and S^h_{yy} are different from 1.0, hence the projected rms-emittances are changed by S^h. Additionally, S^h_{xy} and S^h_{yx} are also different from zero, thus coupling will be imposed to an initially uncoupled beam. Although being non-symplectic, S^h has the determinant of 1.0 preserving the product of the two eigen-emittances.

The beam line being used to determine the periodic solution of an intense coupled beam along a periodic channel is sketched systematically in Fig. 1.

* c.xiao@gsi.de

68th Adv. Beam Dyn. Workshop High-Intensity High-Brightness Hadron Beams HB2023, Geneva, Switzerland JACoW Publishing
ISBN: 978-3-95450-253-0 ISSN: 2673-5571 doi:10.18429/JACoW-HB2023-THA2C3

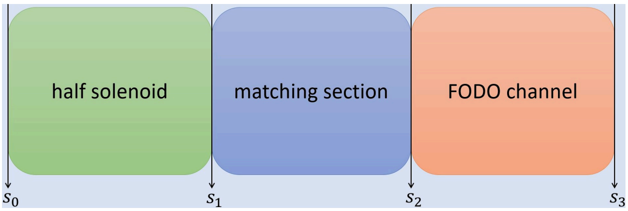

Figure 1: The beam line comprises three parts: (I) effective half solenoid; (II) matching section; (III) regular quadrupole doublet section (twelve cells). Space charge effects are not considered along the first two sections (see text).

At the beginning of the beam line, an uncoupled beam is assumed with beam sigma-matrix $C(s_0)$. The beam matrix at the beginning of the matching section is

$$C(s_1) = S^h \cdot C(s_0) \cdot \left(S^h\right)^{\mathrm{T}} . \quad (5)$$

The matching section is modeled through the symplectic and coupling matrix $\mathfrak{R}$ and hence

$$C(s_2) = \mathfrak{R} \cdot C(s_1) \cdot \mathfrak{R}^{\mathrm{T}} , \quad (6)$$

is the beam matrix at the entrance to the quadrupole channel.

At the entrance to the beam line at s_0, an uncoupled proton beam with an energy of 150 keV/u is assumed. Beam Twiss parameters are set to $\varepsilon_x = \varepsilon_y = 69.90$ mm mrad, $\beta_x = \beta_y = 2$ m/rad, $\alpha_x = 0.250$, and $\alpha_y = -0.275$, while the length of the half solenoid is set to 0.25 m. After transport through this half solenoid the beam matrix (in units of mm and mrad) is

$$C(s_1) = \begin{bmatrix} +133.6 & -8.578 & +2.021 & +124.9 \\ -8.578 & +139.5 & -124.9 & -31.08 \\ +2.021 & -124.9 & +151.4 & +28.22 \\ +124.9 & -31.08 & +28.22 & +154.1 \end{bmatrix} , \quad (7)$$

In order to obtain a periodic solution for a coupled beam, the details of the matching section are not required as seen in the following. However, it is modeled by a transport matrix including 16 elements

$$\mathfrak{R}(m_1, m_2, \ldots, m_{16}) = \begin{bmatrix} m_1 & m_2 & m_3 & m_4 \\ m_5 & m_6 & m_7 & m_8 \\ m_9 & m_{10} & m_{11} & m_{12} \\ m_{13} & m_{14} & m_{15} & m_{16} \end{bmatrix} . \quad (8)$$

Although initially being unknown, the 16 elements must provide for $\det(\mathfrak{R}) = 1.0$ and that $\mathfrak{R}$ is symplectic. For brevity, the set of $m_1, m_2, \ldots, m_{16}$ shall be denoted by $\aleph$.

MODELING OF PERIODIC CHANNEL

For zero current, the effective focusing forces are given solely by the external lattice. The actual beam shape has no influence on them and therefore the periodic solution even for coupled beams may be found analytically. For intense beams instead, defocusing space charge forces depend on the beam shape and orientation in real space. Actually, they depend also on the spatial distribution. However, since modeling of space charge forces using rms-equivalent KV-distributions proofed to work very well for matching purposes, this approach is followed here as well.

The periodic solution meets the condition

$$C(s_2) = \mathfrak{I} \cdot C(s_2) \cdot \mathfrak{I}^{\mathrm{T}} = C(s_2 + \ell) , \quad (9)$$

where ℓ is the length of one cell and the transport matrix from the exit of the solenoid s_1 to the exit of the first cell is

$$\mho(\aleph) = \mathfrak{I} \cdot \mathfrak{R}(\aleph) , \quad (10)$$

where $\mathfrak{I}$ is fully known from the cell of the quadrupole channel.

From first principles, neither the periodic solution is known nor are the elements $\aleph$ that provide for the according matching from the exit of the solenoid s_1 to the entrance of the channel s_2. The iterative procedure to obtain finally both, starts with a guessed initial set $\aleph^i$ that just meets the condition of being symplectic and $\det\left[\mathfrak{R}\left(\aleph^i\right)\right] = 1.0$. It will most likely not meet the condition of the periodic solution, i.e.,

$$\mathfrak{R}\left(\aleph^i\right) \cdot C(s_1) \cdot \mathfrak{R}^{\mathrm{T}}\left(\aleph^i\right) \neq \mho\left(\aleph^i\right) \cdot C(s_1) \cdot \mho^{\mathrm{T}}\left(\aleph^i\right) , \quad (11)$$

hence the beam matrix in front of the channel is different from the one behind the first cell.

With the MATHCAD [7] routine *Minerr*, a set of matching matrix elements $\aleph^0$ for zero beam current can be found, such that the symplectic condition and $\det\left[\mathfrak{R}\left(\aleph^0\right)\right] = 1.0$ is met sharply together with providing periodicity. The routine is dedicated to solve an under-determined system of equations with a defined set boundary conditions.

$$\mathfrak{R}\left(\aleph^0\right) \cdot C(s_1) \cdot \mathfrak{R}^{\mathrm{T}}\left(\aleph^0\right) = \mho\left(\aleph^0\right) \cdot C(s_1) \cdot \mho^{\mathrm{T}}\left(\aleph^0\right) . \quad (12)$$

With $\aleph^0$ being determined, the periodic beam matrix at the beginning of the channel has been calculated as

$$C^0(s_2) = \begin{bmatrix} +158.1 & +0.000 & -76.88 & +95.30 \\ +0.000 & +97.93 & -27.65 & -164.1 \\ +76.88 & -27.65 & +56.66 & +0.000 \\ +95.30 & -164.1 & +0.000 & +438.9 \end{bmatrix} , \quad (13)$$

and it is equal to $C^0(s_2 + \ell)$. As for the case of an uncoupled beam, the periodic solution of the coupled beam features $\alpha_{x,y} = 0$ as expected from the symmetry of the regular cell of the channel. However, the corresponding coupling parameters from combinations of other planes are different from zero due to inter-plane coupling. The zero current transport matrix $\mathfrak{I}\left(\aleph^0\right)$ is independent of the initial beam matrix $C^0(s_2)$ and is determined only by the lattice of the quadrupole channel.

PERIODIC SOLUTION WITH CURRENT

For KV-beams, the electric self-field caused by space charge can be calculated analytically as done by Sacherer [8] for uncoupled beams, i.e., for upright ellipses. In case of

Content from this work may be used under the terms of the CC-BY-4.0 licence (© 2023). Any distribution of this work must maintain attribution to the author(s), title of the work, publisher, and DOI

ISBN: 978-3-95450-253-0 ISSN: 2673-5571 doi:10.18429/JACoW-HB2023-THA2C3

coupling, the ellipse is generally tilted as drawn in Fig. 2. Here, the space charge forces are firstly calculated within the tilted frame. In a second step, these forces are projected into the upright laboratory frame and applied to the beam. They are equivalent to a defocusing quadrupole kick in both planes. The strengths are not equal along both planes but the resulting 4D-transformation is linear and symplectic. Hence it will be modeled by another 4×4 transport matrix $\varkappa$.

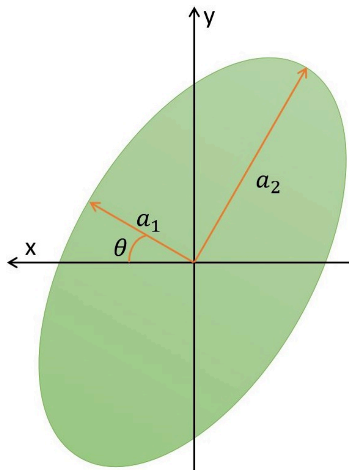

Figure 2: Ellipse of an x-y coupled beam in real space. A_{xy} is the rms-area of the beam, see Eq. (14). Parameters α_{xy} and β_{xy} are its equivalent Twiss parameters defining the ellipse orientation and aspect ratio in real space.

The ellipse is described by its two semi-axes a_1 and a_2 and by the rotation angle θ of a_1 w.r.t. x-axis. Its rms-area is given by

$$A_{xy} = \sqrt{\langle xx\rangle\langle yy\rangle - \langle xy\rangle^2} = a_1 a_2 \,. \tag{14}$$

The above ellipse parameters are calculated from the beam second moments through

$$\beta_{xy} = \frac{\langle xx\rangle}{A_{xy}}\,, \qquad \alpha_{xy} = -\frac{\langle xy\rangle}{A_{xy}}\,, \tag{15}$$

$$\Theta = \frac{1}{2} atan\left[\frac{2 < xy >}{< xx > - < yy >}\right] h = \frac{< xx > + < yy >}{2A_{xy}} \tag{16}$$

$$a_{1,2} = \sqrt{\frac{A_{xy}}{2}}\left(\sqrt{h+1} \pm \sqrt{h-1}\right)\,. \tag{17}$$

The transport matrix $\varkappa$ is calculated from the ellipse geometric parameters and the general beam parameters as

$$\varkappa = R^{-1}(\Theta)\cdot\varkappa^*\cdot R(\Theta)\,, \tag{18}$$

where $\varkappa^*$ is the matrix in the tilted ellipse frame. It reads

$$\varkappa^*_{1,2} = \begin{bmatrix} 1 & 0 \\ \kappa_{1,2}\delta s & 1 \end{bmatrix}, \qquad \varkappa^* = \begin{bmatrix} \varkappa^*_1 & O \\ O & \varkappa^*_2 \end{bmatrix}, \tag{19}$$

with δs being the step size along s between two space charge kicks. $\kappa_{1,2}$ are the respective kick strengths along each semi-axis and are given by

$$\kappa_1 = \frac{\kappa_{sc}}{2a_1(a_1+a_2)}\,, \qquad \kappa_2 = \frac{\kappa_{sc}}{2a_2(a_1+a_2)}\,, \tag{20}$$

from the generalized beam perveance

$$\kappa_{sc} = \frac{qI}{2\pi\epsilon_0 m(\gamma\beta c)^3}\,, \tag{21}$$

with q as particle charge, I as beam current, and β and γ as relativistic factors.

Solutions of the beam matrix along the periodic channel are considered as periodic, if the equation

$$C(s_2) \approx C(s_2+\ell) \tag{22}$$

is fulfilled to very good approximation. Last section presented such a solution $C^0(s_2)$ for zero current. This solution will not hold with beam current being switched on. In order to find a solution that holds even with current, another iterative procedure is applied. It uses the method of determining a matching setting $\aleph$ presented in last section. Additionally, it performs an iterative switching between obtaining the periodic transport matrix from tracking and using it to re-adapt the matching to it.

The iterative procedure starts from the beam moments matrix $C(s_1)$ behind the solenoid being then transported through the matching line $\Re(\aleph^0)$ for zero current. The resulting beam matrix at the entrance to the channel

$$C^0(s_2) = \Re(\aleph^0)\cdot C(s_1)\cdot\Re^T(\aleph^0)\,, \tag{23}$$

is then tracked with high current (10 mA) through one cell. Accordingly, the total transport matrix of the cell $\Im_{sc}(\aleph^0)$ is a result of the tracking procedure for high current. $\Im_{sc}(\aleph^0)$ depends on the current I and on the spatial beam parameters at the entrance of the channel. The 4×4 elements of $\Im_{sc}(\aleph^0)$ are stored for further use. Most likely, $C^0(s_2)$ does not meet the condition of the periodic solution with current, i.e,

$$C^0(s_2) \neq \Im_{sc}(\aleph^0)\cdot\Re(\aleph^0)\cdot C(s_1)\cdot\Re^T(\aleph^0)\cdot\Im^T_{sc}(\aleph^0)\,. \tag{24}$$

However, the cell matrix $\Im_{sc}(\aleph^0)$ is used to re-adapt the matching setting such, that a new matching $\aleph^1$ is found which provides for equal beam matrices before and after transport through the cell matrix $\Im_{sc}(\aleph^0)$

$$C^1(s_2) = \Im_{sc}(\aleph^0)\cdot\Re(\aleph^1)\cdot C(s_1)\cdot\Re^T(\aleph^1)\cdot\Im^T_{sc}(\aleph^0)\,, \tag{25}$$

emphasizing that the above equation uses the stored elements of $\Im_{sc}(\aleph^0)$.

This new matching $\aleph^1$ delivers the beam matrix $C^1(s_2)$ in front of the channel. It is now re-tracked with current through the cell. The tracking will provide a new cell matrix $\Im_{sc}(\aleph^1)$. Again its 4×4 elements are stored to re-adapt the matching to a setting $\aleph^2$ meeting the periodic solution assuming the new matrix $\Im_{sc}(\aleph^1)$ along the channel

$$C^2(s_2) = \Im_{sc}(\aleph^1)\cdot\Re(\aleph^2)\cdot C(s_1)\cdot\Re^T(\aleph^2)\cdot\Im^T_{sc}(\aleph^1)\,. \tag{26}$$

Content from this work may be used under the terms of the CC-BY-4.0 licence (© 2023). Any distribution of this work must maintain attribution to the author(s), title of the work, publisher, and DOI

ISBN: 978-3-95450-253-0 ISSN: 2673-5571 doi:10.18429/JACoW-HB2023-THA2C3

This in turn provides a new beam matrix $C^2(s_2)$ in front of the channel, which changes the transport matrix of the cell to $\mathfrak{I}_{sc}(\aleph^2)$. Continuing this procedure finally converges, i.e., the changes from $\aleph^{n-1}$ to $\aleph^n$ become very small and finally negligible. Accordingly, after a sufficient amount of iterations j, the periodic condition is fulfilled through

$$C^j(s_2) \approx \mathfrak{I}_{sc}(\aleph^j)\cdot\mathfrak{R}(\aleph^j)\cdot C(s_1)\cdot\mathfrak{R}^T(\aleph^j)\cdot\mathfrak{I}_{sc}^T(\aleph^j)\,. \quad (27)$$

The matrix $C^j(s_2)$ contains the periodic beam moments at the entrance to the channel and $\mathfrak{I}_{sc}(\aleph^j)$ is the periodic transport matrix of the cell including current and coupling.

In case of the example presented here, sufficient convergence has been reached at $j = 4$ and the corresponding beam matrix (in units of mm and mrad) is

$$C^4(s_2) = \begin{bmatrix} +153.0 & -0.004 & -86.70 & +0.006 \\ -0.004 & +85.92 & -0.004 & -170.2 \\ -86.70 & -0.004 & +68.44 & +0.019 \\ +0.006 & -170.2 & +0.019 & +431.3 \end{bmatrix}, \quad (28)$$

the corresponding output beam matrix is

$$C^4(s_2+\ell) = \begin{bmatrix} +153.3 & +0.110 & -86.72 & +0.660 \\ +0.110 & +85.71 & -0.215 & -170.2 \\ -86.72 & -0.215 & +68.30 & -0.131 \\ +0.660 & -170.2 & -0.131 & +432.2 \end{bmatrix}, \quad (29)$$

the according transport matrix along the channel (one cell) is determined as

$$\mathfrak{I}_{sc}(\aleph^4) = \begin{bmatrix} +0.476 & +1.263 & +0.126 & +0.022 \\ -0.611 & +0.476 & +0.128 & +0.038 \\ +0.038 & +0.022 & +0.440 & +0.374 \\ +0.128 & +0.126 & -2.148 & +0.441 \end{bmatrix}. \quad (30)$$

The corresponding phase advances are $\mu_x = 61.59°$ and $\mu_y = 63.87°$. These are considerably lower than those of the zero current case of 71.26°.

The corresponding rms-moments along a channel comprising two cells are plotted in Fig. 3. It has been shown that cell-to-cell periodicity of an intense coupled coasting beam can be achieved under the assumption of a KV-distribution.

Results from rms-tracking have been benchmarked with the BEAMPATH code using a Gaussian distribution. Figure 4 plots the respective horizontal and vertical beam envelopes along 10 lattice periods. Very good agreement has been found, hence demonstrating that rms-matching works very well even for coupled beams.

CONCLUSION

It has been shown that cell-to-cell 4D-matching can be achieved for a coupled beam with considerable space charge forces. This has been accomplished by rms-tracking of coupled beams with KV-distribution combined with a dedicated iterative procedure of tracking and re-matching. Hence, it provides a tool for systematic investigations of intense, coupled beam transport along periodic lattices.

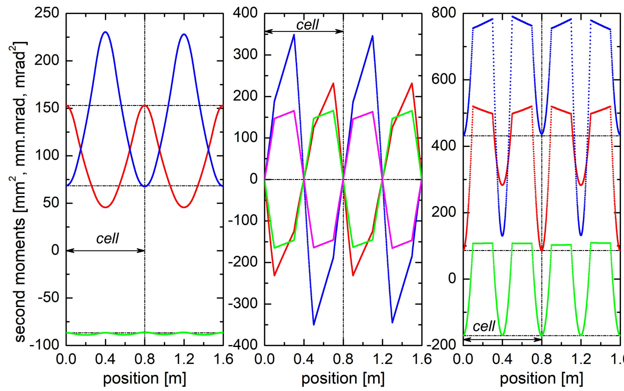

Figure 3: The ten independent rms-moments along the regular quadrupole channel (two cells) for a coupled proton beam with 10 mA. Left: rms-moments $\langle xx\rangle$, $\langle yy\rangle$, and $\langle xy\rangle$ (red, blue, and green); Middle: rms-moments $\langle xx'\rangle$, $\langle yy'\rangle$, $\langle xy'\rangle$, and $\langle x'y\rangle$ (red, blue, green, and magenta); Right: rms-moments $\langle x'x'\rangle$, $\langle y'y'\rangle$, and $\langle x'y'\rangle$ (red, blue, and green).

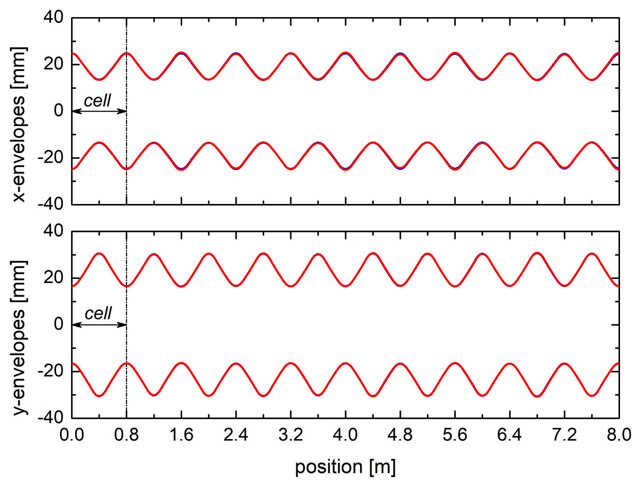

Figure 4: Horizontal and vertical rms-envelopes along the regular quadrupole channel (10 cells) for a coupled proton beam with 10 mA. Blue: rms-tracking, red: BEAMPATH.

REFERENCES

[1] A. Khan, O. Boine-Frankenheim, F. Hug, C. Stoll, "Beam matching with space charge in energy recovery linacs", *Nucl. Instrum. Methods Phys. Res., Sect. A*, vol. 948, p. 162822, 2019. doi:10.1016/j.nima.2019.162822

[2] A. Hoover, N. J. Evans, and J. A. Holmes, "Computation of the matched envelope of the Danilov distribution", *Phys. Rev. Accel. Beams*, vol. 24, p. 044201, 2021. doi:10.1103/PhysRevAccelBeams.24.044201

[3] D. Chernin, "Evolution of rms beam envelopes in transport systems with linear x-y coupling", *Part. Accel.*, vol. 24, pp. 29–44, 1988.

[4] L. Groening, "Concept for controlled transverse emittance transfer within a linac ion beam", *Phys. Rev. Spec. Top. Accel Beams*, vol. 14, p. 064201, 2011. doi:10.1103/PhysRevSTAB.14.064201

[5] C. Xiao, L. Groening, O. Kester, H. Leibrock, M. Maier, and C. Mühle, "Single-knob beam line for transverse emittance

Content from this work may be used under the terms of the CC-BY-4.0 licence (© 2023). Any distribution of this work must maintain attribution to the author(s), title of the work, publisher, and DOI

ISBN: 978-3-95450-253-0 ISSN: 2673-5571 doi:10.18429/JACoW-HB2023-THA2C3

partitioning", *Phys. Rev. Spec. Top. Accel Beams*, vol. 16, p. 044201, 2013. doi.org/10.1103/PhysRevSTAB.16.044201

[6] L. Groening, M. Maier, C. Xiao, L. Dahl, P. Gerhard, O. K. Kester, S. Mickat, H. Vormann, M. Vossberg, and M. Chung, "Experimental proof of adjustable single-knob ion beam emittance partitioning", *Phys. Rev. Lett.*, vol. 113, p. 264802, 2014. doi:10.1103/PhysRevLett.113.264802

[7] PTC Mathcad, https://www.ptc.com/en/engineering-math-software/mathcad/.

[8] F. Sacherer, "RMS envelope equations with space charge", *IEEE Trans. Nucl. Sci.*, vol. 18, no. 3, p. 1105, 1971. doi:10.1109/TNS.1971.4326293

Content from this work may be used under the terms of the CC-BY-4.0 licence (© 2023). Any distribution of this work must maintain attribution to the author(s), title of the work, publisher, and DOI