Published for SISSA by Springer

Received: *February 12, 2026*
Accepted: *April 22, 2026*
Published: *June 17, 2026*

One loop photon-graviton mixing in an electromagnetic field: part 3

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Abstract: Photon-graviton conversion in an electromagnetic field is a well-known prediction of Einstein-Maxwell theory. First discussed at tree-level by Gertsenshtein in 1962, more recently it has been shown to lead to magnetic dichroism starting from one-loop. While previously only two diagrams were assumed to contribute to this one-loop photon-graviton amplitude in a constant electromagnetic field, here we point out the existence of a third one involving a tadpole subdiagram. As shown by H. Gies and one of the authors in 2016 for the pure QED case, such diagrams cannot be omitted in general even though the tadpole formally vanishes. After a short review of the calculation of one-loop photon-graviton amplitudes in the worldline formalism, we use this formalism for a unified calculation of all three diagrams. Although phenomenologically this amplitude is mainly of interest for the case of the spinor loop in a magnetic field, here we will also include the scalar loop and the electric field component, since the computational effort is essentially the same. We show that the tadpole diagram, although contributing to the amplitude, does not contribute to the magnetic dichroism. The gravitational Ward identity provides a useful check.

Keywords: Scattering Amplitudes, Early Universe Particle Physics, Models of Quantum Gravity

ArXiv ePrint: 2601.23279

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https://doi.org/10.1007/JHEP06(2026)175

JHEP06(2026)175

Contents

1 Introduction

Although processes in Einstein-Maxwell theory are presently not of direct experimental relevance, they are of great theoretical interest, and much effort has gone into their study both at the tree-level [1–5] and the one-loop level [6–10].

A particularly well-studied process is photon-graviton conversion in a magnetic field. Einstein-Maxwell theory contains a tree-level vertex involving two photons and one graviton, figure 1.

Replacing one of the photons by an electromagnetic field $F_{\mu\nu}$, one can convert this vertex into the following amplitude for photon-graviton conversion in the field (or the inverse process),

$$\Gamma^{(\text{tree})}(k_0,\epsilon_0;k,\varepsilon;F) = \epsilon_{0\mu\nu}\varepsilon_\alpha \Pi^{\mu\nu,\alpha}_{(\text{tree})}(k;F), \quad \Pi^{\mu\nu,\alpha}_{(\text{tree})}(k;F) = -\frac{i\kappa}{2}C^{\mu\nu,\alpha}\,, \tag{1.1}$$

where k_0,ϵ_0 (k,ε) are the graviton (photon) momentum and polarisation, and the tensor $C^{\mu\nu,\alpha}$ is given by

$$C^{\mu\nu,\alpha} = F^{\mu\alpha}k^\nu + F^{\nu\alpha}k^\mu - (F\cdot k)^\mu\delta^{\nu\alpha} - (F\cdot k)^\nu\delta^{\mu\alpha} + (F\cdot k)^\alpha\delta^{\mu\nu}\,. \tag{1.2}$$

This is well-known, but we give a derivation in appendix A for completeness.

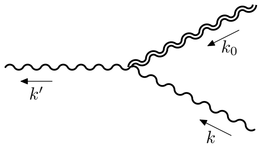

Figure 1. Tree-level photon-photon-graviton vertex in vacuum. Single wiggly lines denote photons, while the double wiggly line denotes the graviton.

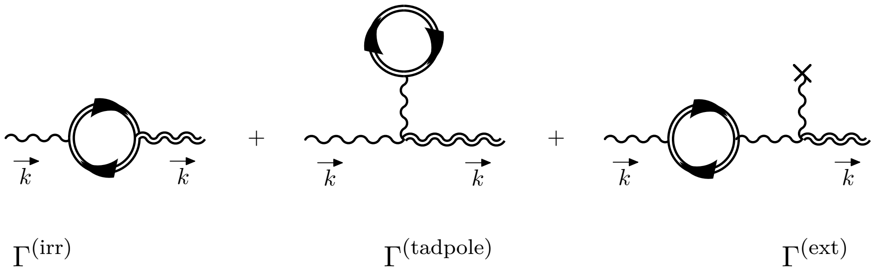

Figure 2. The three one-loop contributions to electromagnetic photon-graviton conversion. Here "irr" and "ext" denote the "irreducible" and "extra" contributions, respectively.

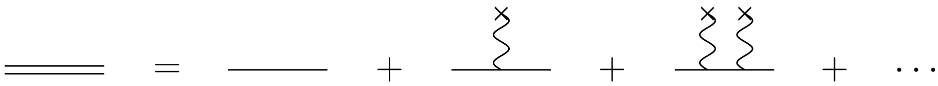

Figure 3. Full scalar or spinor propagator in a constant field.

An important application of this amplitude is the conversion of electromagnetic waves into gravitational waves (and vice-versa) in an external magnetic field, as pointed out by Gertsenshtein as early as 1962 [11]. At the tree level, these processes and the associated photon-graviton conversion have since been studied by many authors [12–19]. However, it is only in 2005 that two of the present authors [20] calculated the one-loop correction to the amplitude (1.1) due to the diagram shown in figure 2 on the left, with either a scalar or spinor loop.

Here we employ the usual straight double-line notation for the full propagator in the constant external field (figure 3), and the wiggly double-line notation to denote the graviton in vacuum.

Using the then novel worldline representation of photon-graviton amplitudes [21–30] they obtained compact parameter integral representations for this amplitude and for both the scalar and spinor loop cases. Those representations are of the same type as the ones that one finds for the better-known one-loop photon vacuum polarisation in a constant field, computed either in the worldline formalism [31, 32] or the Feynman diagram approach (see [33–35] and refs. therein).

Although numerically these one-loop corrections for realistic field strengths are small (suppressed by a factor of α) compared to the tree-level ones, in the subsequent study [36] an important qualitative difference emerged: while the tree-level photon-graviton conversion amplitude in the field gives (contrary to the superficially similar case of photon-axion

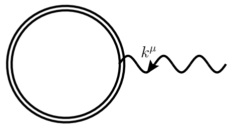

Figure 4. One-photon amplitude in a constant electromagnetic field.

conversion [13, 37–42]) equal conversion rates for both photon polarisations, at the one-loop level this rate starts becoming polarisation-dependent. This leads to dichroism, and further analysis by Ahlers et al. [43], who undertook an exhaustive search for other sources of magnetic dichroism showed that, although magnetic photon-graviton conversion is a very small effect under realistic conditions, remarkably it is still the leading such effect in the whole Standard Model. Their analysis of magnetic dichroism was also more complete in taking the right-hand diagram of figure 2 into account, which contributes to it at the same order in α.

Even so, from their numbers it is clear that this effect will hardly be accessible to experiment in the near future, a conclusion that is consistent with recent detailed feasibility and sensitivity studies of light-induced gravitational effects using interferometric techniques [44, 45]. Nevertheless, this is of course presently the case for all process involving gravitons, and in the long run we would still consider magnetic dichroism due to photon-graviton conversion as a leading contender in the quest for an eventual experimental demonstration of the existence of the graviton. This is because (i) it involves only a single gravitational coupling (ii) it does not require a direct measurement of the graviton itself and (iii) dichroism can be measured with high precision using optical cavities [46–52].

The main purpose of the present paper is to point out the existence of yet another diagram contributing to electromagnetic photon-graviton conversion at the same order, shown in the middle of figure 2, or more precisely its non-vanishing. This requires some explanation. Prior to 2017 it had generally been assumed, and even stated in QED textbooks (see, e.g., [33, 53]) that the one-photon tadpole diagram in a constant electromagnetic field figure 4 vanishes, and therefore also any diagram containing it.

The argument goes as follows:

1. A constant field emits only photons with zero energy-momentum, thus there is a factor of $\delta(k)$.
2. Because of gauge invariance, this diagram in a momentum expansion starts with the term linear in momentum.
3. $\delta(k)k^\mu = 0$.

Since the tadpole thus formally vanishes, it has been assumed for decades that also any diagram containing it can be discarded, for example, the "handcuff" contribution figure 5 to the two-loop Euler-Heisenberg Lagrangian [33, 34].

However, in 2016 H. Gies and one of the authors [54] noted that such diagrams can give finite values because of the infrared divergence of the connecting photon propagator. In dimensional regularization, the key integral is

$$\int d^D k\, \delta^D(k) \frac{k^\mu k^\nu}{k^2} = \frac{\eta^{\mu\nu}}{D}\,. \tag{1.3}$$

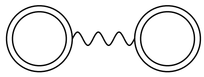

Figure 5. "Handcuff" diagram in a constant field.

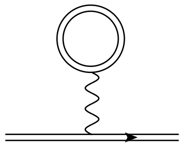

Figure 6. Full scalar or spinor propagator in a constant field.

Applying this integral to the handcuff diagram one finds a non-vanishing result, which can be expressed in the following simple way in terms of the one-loop Euler-Heisenberg Lagrangian [55, 56] (see also [57])

$$\mathcal{L}^{\rm 1PR}_{\rm spin} = \frac{1}{2}\frac{\partial \mathcal{L}^{(EH)}_{\rm spin}}{\partial F^{\mu\nu}}\frac{\partial \mathcal{L}^{(EH)}_{\rm spin}}{\partial F_{\mu\nu}}\,,$$

(the superscript '1PR' stands for "one-particle reducible").

Similarly, [58, 59] found that the one-loop tadpole contribution to the scalar or spinor propagator in a constant field (figure 6) is also non-vanishing, and given by

$$S^{\rm 1PR}(p) = \frac{\partial S(p)}{\partial F_{\mu\nu}}\frac{\partial \mathcal{L}^{(EH)}}{\partial F^{\mu\nu}}\,,$$

where $S(p)$ denotes the tree-level propagator in the field, see also [60].

Below we perform the similar, albeit more complicated, calculation of the middle diagram of figure 2. Moreover, we provide a unified calculation of all three diagrams using the worldline formalism, and show that, although the tadpole diagram contributes to the amplitude, it leaves the magnetic dichroism unchanged. Although our main interest is in the magnetic field and spinor loop case, we will perform all calculations in a general constant field and for both the scalar and spinor loop, since in the worldline formalism this essentially does not imply any extra computational effort. All formulas required to specialize the field to a purely magnetic one (as well as a purely electric or crossed-field one) are given in appendix B.

The organization of the paper is as follows. In section 2 we review the worldline representation of the one-loop photon-graviton amplitudes with a scalar or spinor loop in vacuum, including a short discussion of the gauge and gravitational Ward identities. In section 3 we show how to incorporate an external constant electromagnetic field. Section 4 summarizes the worldline calculation of [20, 36] of the "main" or "irreducible" contributions $\Gamma^{(\rm irr)}$ to these amplitudes, while section 6 analyzes the contribution $\Gamma^{(\rm ext)}$. The central section is 5 where we derive the new "tadpole" contribution $\Gamma^{(\rm tadpole)}$. In section 7 we study the weak-field expansion of the tadpole contribution in a magnetic field. In section 8 we verify the gravitational Ward identities for these amplitudes, which also provides a non-trivial check.

JHEP06(2026)175

Section 9 offers our conclusions. Appendix A establishes our conventions for Einstein-Maxwell theory, while in appendix B we collect some useful formulas involving the worldline Green's functions and determinants in a constant field.

Partial results of this work were presented in [61].

2 One-loop amplitudes with one graviton and N photons

We shortly summarize the worldline approach to the calculation of the one-loop amplitudes involving a scalar or spinor in the loop and one graviton and N photons, first in vacuum [27–30, 62, 63]. These amplitudes have an irreducible contribution, where all the external legs are directly attached to the loop, and a reducible one, where the graviton is attached to one of the photons.

2.1 Irreducible contribution

The irreducible spinor/scalar loop contribution to the one-graviton N-photon amplitude is given by [64]

$$\Gamma^{(\mathrm{irr})}_{\binom{\mathrm{spin}}{\mathrm{scal}}N,1}(k_0,\epsilon_0;\ldots;k_N,\varepsilon_N)=\binom{-2}{1}(-ie)^N\left(-\frac{\kappa}{4}\right)\int_0^\infty\frac{dT}{T}\,e^{-m^2T}(4\pi T)^{-\frac{D}{2}}$$
$$\times\Big\langle V^G_{\binom{\mathrm{spin}}{\mathrm{scal}}}[k_0,\epsilon_0]\,V^\gamma_{\binom{\mathrm{spin}}{\mathrm{scal}}}[k_1,\varepsilon_1]\cdots V^\gamma_{\binom{\mathrm{spin}}{\mathrm{scal}}}[k_N,\varepsilon_N]\Big\rangle. \tag{2.1}$$

Here e and κ are the electromagnetic and gravitational couplings, m the mass and T the proper-time of the loop scalar or spinor. k_0 and ϵ_0 denote the graviton momentum and polarization tensor, the remaining k_i's and ε_i's the photon ones. V^G and V^γ denote the graviton and photon vertex operators,

$$V^\gamma_{\mathrm{scal}}[k,\varepsilon]=\int_0^T d\tau\,\varepsilon\cdot\dot{x}(\tau)\;\mathrm{e}^{ik\cdot x(\tau)}\,, \tag{2.2}$$
$$V^\gamma_{\mathrm{spin}}[k,\varepsilon]=\int_0^T d\tau\,\Big[\varepsilon\cdot\dot{x}(\tau)-i\psi(\tau)\cdot f\cdot\psi(\tau)\Big]\;\mathrm{e}^{ik\cdot x(\tau)}\,, \tag{2.3}$$
$$V^G_{\mathrm{scal}}[k_0,\epsilon_0]=\epsilon_{0\mu\nu}\int_0^T d\tau\Big[\dot{x}^\mu(\tau)\dot{x}^\nu(\tau)+a^\mu(\tau)a^\nu(\tau)+b^\mu(\tau)c^\nu(\tau)\Big]\;\mathrm{e}^{ik_0\cdot x(\tau)}\,, \tag{2.4}$$
$$V^G_{\mathrm{spin}}[k_0,\epsilon_0]=\epsilon_{0\mu\nu}\int_0^T d\tau\Big[\dot{x}^\mu(\tau)\dot{x}^\nu(\tau)+a^\mu(\tau)a^\nu(\tau)+b^\mu(\tau)c^\nu(\tau)$$
$$+\psi^\mu(\tau)\dot{\psi}^\nu(\tau)+\alpha^\mu(\tau)\alpha^\nu(\tau)+i\dot{x}^\mu(\tau)\psi^\nu(\tau)\psi(\tau)\cdot k\Big]\;\mathrm{e}^{ik_0\cdot x(\tau)}\,, \tag{2.5}$$

and $f_{\mu\nu}=k_\mu\varepsilon_\nu-k_\nu\varepsilon_\mu$ the linearized photon field-strength tensor. The Wick contractions $\langle\cdots\rangle$ are to be performed using the set of worldline correlators

$$\langle x^\mu(\tau)x^\nu(\tau')\rangle=-G_B(\tau,\tau')\,, \tag{2.6}$$
$$\langle\psi^\mu(\tau)\psi^\nu(\tau')\rangle=\frac{1}{2}G_F(\tau,\tau')\,, \tag{2.7}$$
$$\langle a^\mu(\tau)a^\nu(\tau')\rangle=2\delta(\tau-\tau')\delta^{\mu\nu}\,, \tag{2.8}$$
$$\langle b^\mu(\tau)c^\nu(\tau')\rangle=-4\delta(\tau-\tau')\delta^{\mu\nu}\,, \tag{2.9}$$
$$\langle\alpha^\mu(\tau)\alpha^\nu(\tau')\rangle=2\delta(\tau-\tau')\delta^{\mu\nu}\,, \tag{2.10}$$

with the worldline Green's functions

$$G_B(\tau,\tau') =\mid \tau-\tau' \mid - \frac{(\tau-\tau')^2}{T}\,, \tag{2.11}$$

$$G_F(\tau,\tau') = \mathrm{sgn}(\tau-\tau')\,. \tag{2.12}$$

2.2 Reducible contribution

The reducible contribution to the N-photon/one graviton amplitude, where the graviton is attached to one of the photons, usually would be constructed by sewing together the N-photon amplitude and the photon-photon-graviton vertex of figure 1. For the analogous case of the scalar tree-level propagator dressed with N photons and one graviton, $D^{(\mathrm{red})pp'}_{N,1}$, it was shown in [63] that the effect of this sewing can be expressed by the following replacement rule

$$D^{(\mathrm{red})pp'}_{N,1}(k_0,\epsilon_0;\ldots;k_N,\varepsilon_N) = \sum_{i=1}^{N} D^{pp'}_{N,0}(k_1,\varepsilon_1;\ldots;k_i+k_0,\upsilon_i;\ldots;k_N,\varepsilon_N)\,,$$

with an effective photon polarization vector

$$\upsilon_i \equiv -\kappa\frac{\{\epsilon_0,f_i\}\cdot(k_i+k_0)}{(k_i+k_0)^2} = -\kappa\frac{\epsilon_0\cdot f_i\cdot k_0 + f_i\cdot\epsilon_0\cdot k_i}{2k_0\cdot k_i}\,. \tag{2.13}$$

Here we observe that the same replacement rule applies as well for the one-loop N-photon/one graviton amplitudes with either a scalar or a spinor loop. For the scalar loop this is obvious from the fact that our derivation of that rule in [63] did not make use at all of the boundary conditions imposed on the scalar path integral. But neither did we use the specific form of the scalar vertex operator, and it is easy to see that changing the scalar vertex operator into a spinor one will not make a difference here. Thus, without further ado, we can write down the reducible part of the spinor/scalar loop contribution to the one-graviton N-photon amplitude as

$$\Gamma^{(\mathrm{red})}_{\binom{\mathrm{spin}}{\mathrm{scal}}N,1}(k_0,\epsilon_0;\ldots;k_N,\varepsilon_N) = \sum_{i=1}^{N}\Gamma_{\binom{\mathrm{spin}}{\mathrm{scal}}N,0}(k_1,\varepsilon_1;\ldots;k_i+k_0,\upsilon_i;\ldots;k_N,\varepsilon_N)\,, \tag{2.14}$$

where υ_i^μ is the same as in (2.13).

3 Inclusion of a constant electromagnetic field

3.1 General procedure

The generalisation of the formalism to the case of the photon-graviton amplitudes in a constant electromagnetic field is straightforward and proceeds essentially as in the purely electromagnetic case [23, 25, 26, 32, 67–69]: the master formulas (2.1) acquire an additional determinant factor

$$\det{}^{\frac{1}{2}}\left[\frac{\mathcal{Z}}{\tan\mathcal{Z}}\right] \qquad \text{(Spinor loop)}\,, \tag{3.1}$$

$$\det{}^{\frac{1}{2}}\left[\frac{\mathcal{Z}}{\sin\mathcal{Z}}\right] \qquad \text{(Scalar loop)}\,, \tag{3.2}$$

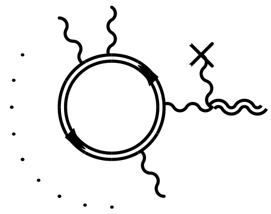

Figure 7. Field-induced type of reducible contributions.

($\mathcal{Z}_{\mu\nu} \equiv eF_{\mu\nu}T$), and the worldline Green's functions G_B, G_F have to be replaced by their constant-field versions $\mathcal{G}_B, \mathcal{G}_F$,

$$\mathcal{G}_B(\tau,\tau') = \frac{T}{2\mathcal{Z}}\left(\frac{\mathrm{e}^{-i\dot{G}_B(\tau,\tau')\mathcal{Z}} - \cos\mathcal{Z}}{\sin\mathcal{Z}} + i\dot{G}_B(\tau,\tau')\right),$$
$$\mathcal{G}_F(\tau,\tau') = G_F(\tau,\tau')\frac{\mathrm{e}^{-i\mathcal{Z}\dot{G}_B(\tau,\tau')}}{\cos\mathcal{Z}}. \tag{3.3}$$

Since these Green's functions have a non-trivial Lorentz structure, the Wick-contraction rules (2.6), (2.7) generalise in the form

$$\langle x^\mu(\tau)x^\nu(\tau')\rangle_F = -\mathcal{G}_B^{\mu\nu}(\tau,\tau'), \tag{3.4}$$
$$\langle \psi^\mu(\tau)\psi^\nu(\tau')\rangle_F = \frac{1}{2}\mathcal{G}_F^{\mu\nu}(\tau,\tau'), \tag{3.5}$$

with the Wick-contraction rules for the ghosts unchanged.

Thus the vacuum master formula (2.1) directly generalises to the constant-field background as

$$\Gamma^{(\mathrm{irr})}_{\binom{\mathrm{spin}}{\mathrm{scal}}N,1}(k_0,\epsilon_0;\ldots;k_N,\varepsilon_N;F) = \binom{-2}{1}(-ie)^N\left(-\frac{\kappa}{4}\right)\int_0^\infty\frac{dT}{T}e^{-m^2T}(4\pi T)^{-\frac{D}{2}}\det{}^{\frac{1}{2}}\left[\begin{pmatrix}\mathcal{Z}/\tan\mathcal{Z}\\ \mathcal{Z}/\sin\mathcal{Z}\end{pmatrix}\right]$$
$$\times\left\langle V^G_{\binom{\mathrm{spin}}{\mathrm{scal}}}[k_0,\epsilon_0]V^\gamma_{\binom{\mathrm{spin}}{\mathrm{scal}}}[k_1,\varepsilon_1]\cdots V^\gamma_{\binom{\mathrm{spin}}{\mathrm{scal}}}[k_N,\varepsilon_N]\right\rangle_F. \tag{3.6}$$

The rule (2.14) for the construction of the reducible contributions, where the graviton gets attached to one of the photons, is also not affected by the presence of the external field.

At the one-graviton level, the only really new aspect arising from the presence of the external field is that there is now a second type of reducible contributions, since the graviton can now be attached also to one of the photons emitted by the external field, see figure 7.

It is easy to see that the contribution of this diagram, to be called $\Gamma^{(\mathrm{ext})}_{N,1}$, can be obtained from the purely photonic amplitude with $N+1$ photons in the field by the following modification of (2.14),

$$\Gamma^{(\mathrm{ext})}_{\binom{\mathrm{spin}}{\mathrm{scal}}N,1}(k_0,\epsilon_0;\ldots;k_N,\varepsilon_N;F) = \Gamma_{\binom{\mathrm{spin}}{\mathrm{scal}}N+1,0}(k_0,\upsilon_F;k_1,\varepsilon_1;\ldots;\ldots;k_N,\varepsilon_N;F), \tag{3.7}$$

where now

$$\upsilon_F \equiv i\kappa\frac{\{\epsilon_0,F\}\cdot k_0}{k_0^2}. \tag{3.8}$$

3.2 The photon tadpole

For starters, let us work out (3.6) for the pure QED case with one and two photons. For the tadpole (figure 4) with a scalar loop, this yields

$$\Gamma_{\rm scal}(k,\varepsilon;F) = -e(2\pi)^D\delta^D(k)\int_0^\infty \frac{dT}{T}(4\pi T)^{-\frac{D}{2}}e^{-m^2T}{\rm det}^{\frac12}\left[\frac{\mathcal{Z}}{\sin\mathcal{Z}}\right] \times\int_0^T d\tau\,\varepsilon\cdot\dot{\mathcal{G}}_B(\tau,\tau)\cdot k\exp\left[\frac12 k\cdot\mathcal{G}_B(\tau,\tau)\cdot k\right]. \tag{3.9}$$

Using the coincidence limits (B.6), (B.7) this can be written more explicitly as

$$\Gamma_{\rm scal}(k,\varepsilon;F) = -ie(2\pi)^D\delta^D(k)\int_0^\infty dT(4\pi T)^{-\frac{D}{2}}e^{-m^2T}{\rm det}^{\frac12}\left[\frac{\mathcal{Z}}{\sin\mathcal{Z}}\right] \times\varepsilon\cdot\left(\cot\mathcal{Z}-\frac{1}{\mathcal{Z}}\right)\cdot k\exp\left[\frac{T}{4}k\cdot\left(\frac{\cot\mathcal{Z}}{\mathcal{Z}}-\frac{1}{\mathcal{Z}^2}\right)\cdot k\right]. \tag{3.10}$$

To get the corresponding quantity for spinor QED, we have to include the spin part of the photon vertex operator, change the determinant factor, and to supply the usual global factor of -2. This yields

$$\Gamma_{\rm spin}(k,\varepsilon;F) = 2ie(2\pi)^D\delta^D(k)\int_0^\infty dT(4\pi T)^{-\frac{D}{2}}e^{-m^2T}{\rm det}^{\frac12}\left[\frac{\mathcal{Z}}{\tan\mathcal{Z}}\right] \times\varepsilon\cdot\left(\cot\mathcal{Z}+\tan\mathcal{Z}-\frac{1}{\mathcal{Z}}\right)\cdot k\exp\left[\frac{T}{4}k\cdot\left(\frac{\cot\mathcal{Z}}{\mathcal{Z}}-\frac{1}{\mathcal{Z}^2}\right)\cdot k\right]. \tag{3.11}$$

Note that, as mentioned in the introduction, the expressions (3.10), (3.11) formally vanish on account of the factor $\delta^D(k)\,k^\mu$.

3.3 The vacuum-polarization tensor in a constant field

For $N=2$ in the scalar case we find the integrand

$$\Big\langle V^\gamma_{\rm scal}[k_1,\varepsilon_1]V^\gamma_{\rm scal}[k_2,\varepsilon_2]\Big\rangle_F = \Big[\varepsilon_1\cdot\ddot{\mathcal{G}}_{B12}\cdot\varepsilon_2-\varepsilon_1\cdot\dot{\mathcal{G}}_{B1i}\cdot k_i\varepsilon_2\cdot\dot{\mathcal{G}}_{B2j}\cdot k_j\Big]\,{\rm e}^{\frac12 k_i\cdot\mathcal{G}_{Bij}\cdot k_j}, \tag{3.12}$$

where summation over $i,j=1,2$ is understood. We use momentum conservation to write $k=k_1=-k_2$. Removing the second derivative in the first term by an IBP in τ_1, the integrand becomes

$$\Big[\varepsilon_1\cdot\dot{\mathcal{G}}_{B12}\cdot\varepsilon_2 k\cdot\dot{\mathcal{G}}_{B12}\cdot k+\varepsilon_1\cdot\Big(\dot{\mathcal{G}}_{B12}-\dot{\mathcal{G}}_B\Big)\cdot k\varepsilon_2\cdot\Big(\dot{\mathcal{G}}_{B21}-\dot{\mathcal{G}}_B\Big)\cdot k\Big]\,{\rm e}^{-k\cdot(\mathcal{G}_{B12}-\mathcal{G}_B)\cdot k}, \tag{3.13}$$

where we have introduced the further notation of indicating the constant coincidence limit of a Green's function by the omission of its argument, $\dot{\mathcal{G}}_B\equiv\dot{\mathcal{G}}_{Bii}$ etc. The content of the brackets then turns into $\varepsilon_{1\mu}I^{\mu\nu}_{\rm scal}\varepsilon_{2\nu}$,

$$I^{\mu\nu}_{\rm scal} = \dot{\mathcal{G}}^{\mu\nu}_{B12}k\cdot\dot{\mathcal{G}}_{B12}\cdot k+\Big(\dot{\mathcal{G}}_{B12}-\dot{\mathcal{G}}_B\Big)^{\mu\lambda}\Big(\dot{\mathcal{G}}_{B21}-\dot{\mathcal{G}}_B\Big)^{\nu\kappa}k^\kappa k^\lambda. \tag{3.14}$$

Next we would like to use the fact that the integrand contains terms which integrate to zero due to antisymmetry under the exchange $\tau_1\leftrightarrow\tau_2$. Thus we decompose the constant-field worldline Green's function as

$$\mathcal{G}_{B12} = \mathcal{S}_{B12}+\mathcal{A}_{B12}, \tag{3.15}$$

where $\mathcal{S}_{Bij}$ denotes the even ($\mathcal{S}_{Bii} \equiv \mathcal{S}_B$) and $\mathcal{A}_{Bij}$ the odd part ($\mathcal{A}_{Bii} \equiv \mathcal{A}_B$) as a power series in F. Only the Lorentz even part of $\mathcal{G}_{Bij}$ contributes in the exponent,

$$k \cdot (\mathcal{G}_{B12} - \mathcal{G}_B) \cdot k = k \cdot (\mathcal{S}_{B12} - \mathcal{S}_B) \cdot k \equiv Tk \cdot \Phi_{12} \cdot k\,. \tag{3.16}$$

$I^{\mu\nu}_{\rm scal}$ turns, after decomposing all factors of $\dot{\mathcal{G}}_{Bij}$ as above, and deleting all τ — odd terms, into

$$\tilde{I}^{\mu\nu}_{\rm scal} \equiv \Big\{\dot{\mathcal{S}}^{\mu\nu}_{B12}\dot{\mathcal{S}}^{\kappa\lambda}_{B12} - \dot{\mathcal{S}}^{\mu\lambda}_{B12}\dot{\mathcal{S}}^{\nu\kappa}_{B12} + \Big(\dot{\mathcal{A}}_{B12} - \dot{\mathcal{A}}_B\Big)^{\mu\lambda}\Big(\dot{\mathcal{A}}_{B12} - \dot{\mathcal{A}}_B\Big)^{\nu\kappa}\Big\}k^\kappa k^\lambda\,. \tag{3.17}$$

In this way we obtain the following integral representations for the dimensionally regularized constant-field vacuum polarization tensor in scalar QED:

$$\Pi^{\mu\nu}_{\rm scal}(k;F) = -\frac{e^2}{(4\pi)^{\frac{D}{2}}}\int_0^\infty \frac{dT}{T}T^{2-\frac{D}{2}}e^{-m^2T}{\rm det}^{\frac12}\left[\frac{\mathcal{Z}}{\sin\mathcal{Z}}\right]\int_0^1 du_1\,\tilde{I}^{\mu\nu}_{\rm scal}\,{\rm e}^{-Tk\cdot\Phi_{12}\cdot k}\,. \tag{3.18}$$

As usual we have rescaled to the unit circle and set $u_2 = 0$.

For spinor QED, the formula corresponding to (3.14) is [32]

$$\begin{aligned} I^{\mu\nu}_{\rm spin} &= \dot{\mathcal{G}}^{\mu\nu}_{B12}k\cdot\dot{\mathcal{G}}_{B12}\cdot k - \mathcal{G}^{\mu\nu}_{F12}k\cdot\mathcal{G}_{F12}\cdot k \\ &\quad - \Big[\Big(\dot{\mathcal{G}}_B - \mathcal{G}_F - \dot{\mathcal{G}}_{B12}\Big)^{\mu\lambda}\Big(\dot{\mathcal{G}}_{B21} - \dot{\mathcal{G}}_B + \mathcal{G}_F\Big)^{\nu\kappa} + \mathcal{G}^{\mu\lambda}_{F12}\mathcal{G}^{\nu\kappa}_{F21}\Big]k^\kappa k^\lambda \end{aligned} \tag{3.19}$$

and (3.17) generalizes to

$$\begin{aligned} \tilde{I}^{\mu\nu}_{\rm spin} \equiv \Big\{&\Big(\dot{\mathcal{S}}^{\mu\nu}_{B12}\dot{\mathcal{S}}^{\kappa\lambda}_{B12} - \dot{\mathcal{S}}^{\mu\lambda}_{B12}\dot{\mathcal{S}}^{\nu\kappa}_{B12}\Big) - \Big(\mathcal{S}^{\mu\nu}_{F12}\mathcal{S}^{\kappa\lambda}_{F12} - \mathcal{S}^{\mu\lambda}_{F12}\mathcal{S}^{\nu\kappa}_{F12}\Big) \\ &+ \Big(\dot{\mathcal{A}}_{B12} - \dot{\mathcal{A}}_B + \mathcal{A}_F\Big)^{\mu\lambda}\Big(\dot{\mathcal{A}}_{B12} - \dot{\mathcal{A}}_B + \mathcal{A}_F\Big)^{\nu\kappa} \\ &- \mathcal{A}^{\mu\lambda}_{F12}\mathcal{A}^{\nu\kappa}_{F12}\Big\}k^\kappa k^\lambda\,. \end{aligned} \tag{3.20}$$

Taking the global factor of -2 and the change of the field-dependent determinant factor (3.2) into account, (3.18) generalizes to

$$\Pi^{\mu\nu}_{\rm spin}(k;F) = 2\frac{e^2}{(4\pi)^{\frac{D}{2}}}\int_0^\infty \frac{dT}{T}T^{2-\frac{D}{2}}e^{-m^2T}{\rm det}^{\frac12}\left[\frac{\mathcal{Z}}{\tan\mathcal{Z}}\right]\int_0^1 du_1\,\tilde{I}^{\mu\nu}_{\rm spin}\,{\rm e}^{-Tk\cdot\Phi_{12}\cdot k}\,. \tag{3.21}$$

As is well-known, the constant field vacuum polarization tensors contain the UV divergences of the ordinary vacuum polarization tensors, but no further ones, so that on-shell renormalization can be performed by subtraction of the integrand at zero field strength and momentum. In this way we find for the renormalized vacuum polarization tensors (indicating the renormalisation by a 'bar') [32]

$$\begin{aligned} \bar{\Pi}^{\mu\nu}_{\rm scal}(k;F) = -\frac{e^2}{16\pi^2}\int_0^\infty\frac{dT}{T}e^{-m^2T}\int_0^1 du_1\Big\{&{\rm det}^{\frac12}\left[\frac{\mathcal{Z}}{\sin\mathcal{Z}}\right]\tilde{I}^{\mu\nu}_{\rm scal}\,{\rm e}^{-Tk\cdot\Phi_{12}\cdot k} \\ &- (\delta^{\mu\nu}k^2 - k^\mu k^\nu)(1-2u_1)^2\Big\}\,, \end{aligned} \tag{3.22}$$

$$\begin{aligned} \bar{\Pi}^{\mu\nu}_{\rm spin}(k;F) = \frac{e^2}{8\pi^2}\int_0^\infty\frac{dT}{T}e^{-m^2T}\int_0^1 du_1\Big\{&{\rm det}^{\frac12}\left[\frac{\mathcal{Z}}{\tan\mathcal{Z}}\right]\tilde{I}^{\mu\nu}_{\rm spin}\,{\rm e}^{-Tk\cdot\Phi_{12}\cdot k} \\ &+ 4(\delta^{\mu\nu}k^2 - k^\mu k^\nu)u_1(1-u_1)\Big\}\,. \end{aligned} \tag{3.23}$$

The remaining u_1 — integral can be brought into a more standard form by a transformation of variables $v = \dot{G}_{B12} = 1 - 2u_1$.

The matrix decomposition formulas of appendix B can be used to write these integral representations more explicitly (for the generic case of parallel electric and magnetic fields this has been done in [32]).

4 Irreducible contribution to electromagnetic photon-graviton conversion

We now turn to applying this formalism to the electromagnetic photon-graviton conversion amplitude, starting with the irreducible contribution $\Gamma^{(\mathrm{irr})}_{\mathrm{spin}}$. Using the machinery developed above, its worldline representation can be written as

$$\Gamma^{(\mathrm{irr})}_{\mathrm{spin}}(k_0, \epsilon_0; k, \varepsilon; F) = -2(-ie)\left(-\frac{\kappa}{4}\right)\int_0^\infty \frac{dT}{T}\, e^{-m^2T}(4\pi T)^{-\frac{D}{2}}\mathrm{det}^{-\frac{1}{2}}\left[\frac{\tan(\mathcal{Z})}{\mathcal{Z}}\right] \times \left\langle V^G_{\mathrm{spin}}[k_0, \epsilon_0] V^\gamma_{\mathrm{spin}}[k, \varepsilon]\right\rangle_F . \tag{4.1}$$

Performing the Wick contractions according to the rules for the constant-field case, eqs. (3.4) and (3.5), stripping off ϵ_0, ε as in (1.1), and eliminating k_0 through $k_0 = -k$, the result can be written as

$$\Pi^{\mu\nu,\alpha}_{\mathrm{spin}}(k; F) = -\frac{e\kappa}{2(4\pi)^{\frac{D}{2}}}\int_0^\infty \frac{dT}{T}\, e^{-m^2T}T^{-\frac{D}{2}}\,\mathrm{det}^{-\frac{1}{2}}\left[\frac{\tan\mathcal{Z}}{\mathcal{Z}}\right] \times \int_0^T d\tau_1 \int_0^T d\tau_2\; \mathrm{e}^{-k\cdot\bar{\mathcal{G}}_{B12}\cdot k}\left(J^{\mu\nu,\alpha}_{\mathrm{scal}} + J^{(\mu\nu),\alpha}_{\psi}\right), \tag{4.2}$$

where in $J^{\mu\nu,\alpha}_{\mathrm{scal}}$ we have collected all terms that come from the purely orbital part of the vertex operators, and $J^{(\mu\nu),\alpha}_{\psi} \equiv \frac{1}{2}(J^{(\mu\nu),\alpha}_{\psi} + J^{(\nu\mu),\alpha}_{\psi})$ contains all the terms involving the spin of the loop fermion. Here $\bar{\mathcal{G}}_{B12} \equiv \mathcal{G}_{B12} - \mathcal{G}_B$ is the modified Green's function obtained by subtracting its coincidence limit. Using the symmetry properties (B.5) and (B.37), the result can be written as

$$\begin{aligned} J^{\mu\nu,\alpha}_{\mathrm{scal}} = {} & \left(\ddot{\mathcal{G}}^{\mu\nu}_{B11} - 2\delta_{11}\delta^{\mu\nu}\right)\left(k\cdot\dot{\bar{\mathcal{G}}}_{B12}\right)^\alpha + \left[\ddot{\mathcal{G}}^{\mu\alpha}_{B12}\left(\dot{\bar{\mathcal{G}}}_{B12}\cdot k\right)^\nu + \left(\mu \leftrightarrow \nu\right)\right] \\ & - \left(\dot{\bar{\mathcal{G}}}_{B12}\cdot k\right)^\mu\left(\dot{\bar{\mathcal{G}}}_{B12}\cdot k\right)^\nu\left(k\cdot\dot{\bar{\mathcal{G}}}_{B12}\right)^\alpha , \end{aligned} \tag{4.3}$$

$$\begin{aligned} J^{\mu\nu,\alpha}_{\psi} = {} & \left[\dot{\mathcal{G}}^{\mu\nu}_{F11} - 2\delta_{11}\delta^{\mu\nu} + \left(\mathcal{G}_{F11}\cdot k\right)^\nu\left(\dot{\bar{\mathcal{G}}}_{B12}\cdot k\right)^\mu\right]\left(\mathcal{G}_{F22}\cdot k\right)^\alpha \\ & - \left(\mathcal{G}_{F12}\cdot k\right)^\mu \dot{\mathcal{G}}^{\nu\alpha}_{F12} + \mathcal{G}^{\mu\alpha}_{F12}\left(\dot{\mathcal{G}}_{F12}\cdot k\right)^\nu \\ & + \left[-\mathcal{G}^{\nu\alpha}_{F12}\left(k\cdot\mathcal{G}_{F12}\cdot k\right) + \left(\mathcal{G}_{F12}\cdot k\right)^\nu\left(k\cdot\mathcal{G}_{F12}\right)^\alpha\right]\left(\dot{\bar{\mathcal{G}}}_{B12}\cdot k\right)^\mu \\ & - \left(\dot{\mathcal{G}}^{\mu\nu}_{F11} - 2\delta_{11}\delta^{\mu\nu}\right)\left(k\cdot\dot{\bar{\mathcal{G}}}_{B12}\right)^\alpha \\ & - \left[\ddot{\mathcal{G}}^{\mu\nu}_{B11} - 2\delta_{11}\delta^{\mu\nu} - \left(\dot{\bar{\mathcal{G}}}_{B12}\cdot k\right)^\mu\left(\dot{\bar{\mathcal{G}}}_{B12}\cdot k\right)^\nu\right]\left(\mathcal{G}_{F22}\cdot k\right)^\alpha \\ & + \left[\ddot{\mathcal{G}}^{\mu\alpha}_{B12} - \left(\dot{\bar{\mathcal{G}}}_{B12}\cdot k\right)^\mu\left(k\cdot\dot{\bar{\mathcal{G}}}_{B12}\right)^\alpha\right]\left(\mathcal{G}_{F11}\cdot k\right)^\nu . \end{aligned} \tag{4.4}$$

It will be useful to add to $J^{\mu\nu,\alpha}_{\rm scal}\,{\rm e}^{-k\cdot\overline{\mathcal{G}}_{B12}\cdot k}$ the total derivative term

$$-\frac{1}{2}\frac{\partial}{\partial\tau_1}\Big[\bar{\dot{\mathcal{G}}}^{\mu\alpha}_{B12}\Big(\bar{\dot{\mathcal{G}}}_{B12}\cdot k\Big)^{\nu}\,{\rm e}^{-k\cdot\overline{\mathcal{G}}_{B12}\cdot k}+\Big(\mu\leftrightarrow\nu\Big)\Big]. \tag{4.5}$$

We can then replace $J^{\mu\nu,\alpha}_{\rm scal}+J^{(\mu\nu),\alpha}_{\psi}$ by

$$J^{\mu\nu,\alpha}_{\rm spin}=J^{\mu\nu,\alpha}_{\rm spin,1}+J^{(\mu\nu),\alpha}_{\rm spin,2}+J^{(\mu\nu),\alpha}_{\rm spin,3} \tag{4.6}$$

where

$$\begin{aligned}
J^{\mu\nu,\alpha}_{\rm spin,1}&=-\Big(\ddot{\mathcal{G}}^{\mu\nu}_{B11}-\dot{\mathcal{G}}^{\mu\nu}_{F11}\Big)\Big[(\bar{\dot{\mathcal{G}}}_{B21}+\mathcal{G}_{F22})\cdot k\Big]^{\alpha},\\
J^{\mu\nu,\alpha}_{\rm spin,2}&=-\bar{\dot{\mathcal{G}}}^{\mu\alpha}_{B12}\Big(\ddot{\mathcal{G}}_{B12}\cdot k\Big)^{\nu}+\mathcal{G}^{\mu\alpha}_{F12}\Big(\dot{\mathcal{G}}_{F12}\cdot k\Big)^{\nu}+\ddot{\mathcal{G}}^{\nu\alpha}_{B12}\Big[(\bar{\dot{\mathcal{G}}}_{B12}+\mathcal{G}_{F11})\cdot k\Big]^{\mu}\\
&\quad-\dot{\mathcal{G}}^{\nu\alpha}_{F12}\Big(\mathcal{G}_{F12}\cdot k\Big)^{\mu},\\
J^{\mu\nu,\alpha}_{\rm spin,3}&=\Big(\bar{\dot{\mathcal{G}}}_{B12}\cdot k\Big)^{\mu}\Big\{\Big[(\bar{\dot{\mathcal{G}}}_{B12}+\mathcal{G}_{F11})\cdot k\Big]^{\nu}\Big[(\bar{\dot{\mathcal{G}}}_{B21}+\mathcal{G}_{F22})\cdot k\Big]^{\alpha}\\
&\quad-\Big(\mathcal{G}_{F12}k\Big)^{\nu}\Big(\mathcal{G}_{F21}k\Big)^{\alpha}+\bar{\dot{\mathcal{G}}}^{\nu\alpha}_{B12}k\cdot\dot{\mathcal{G}}_{B12}\cdot k-\mathcal{G}^{\nu\alpha}_{F12}k\cdot\mathcal{G}_{F12}\cdot k\Big\}.
\end{aligned} \tag{4.7}$$

Before proceeding further, let us use this integral representation to analyse the general structure of this amplitude. Although our calculation is nonperturbative in the external field, we can use the series expansions of $\mathcal{G}_B,\dot{\mathcal{G}}_B,\ddot{\mathcal{G}}_B$ and $\mathcal{G}_F,\dot{\mathcal{G}}_F$ in the field, whose first few terms are given in (B.9)–(B.11) and (B.40), (B.41), to compute the amplitude involving a given number of interactions with the field. It is then immediately seen that this amplitude is nonzero only if this number of interactions is odd, since otherwise the $\tau_{1,2}$ integrations vanish by antisymmetry.

The leading term in this expansion in the background field is UV divergent. In dimensional regularisation, this divergence is [20]

$$\Pi^{\mu\nu,\alpha}_{\rm spin,div}(k;F)=-\frac{4}{3}\frac{ie^2\kappa}{(4\pi)^2}\frac{1}{D-4}C^{\mu\nu,\alpha}(k)\,, \tag{4.8}$$

where $C^{\mu\nu,\alpha}$ is the same tensor that appeared in the tree-level amplitude, eq. (1.2). This was to be expected since the amplitude must be multiplicatively renormalizable (although quantum gravity is not a renormalizable theory, amplitudes in Einstein-Maxwell theory involving gravitational fields and gravitons only externally are still renormalizable by power counting). Renormalization is performed by subtracting the amplitude at zero field strength and zero momentum, leading to the following form of the renormalized amplitude $\bar{\Pi}$:

$$\begin{aligned}
\bar{\Pi}^{\mu\nu,\alpha}_{\rm spin}(k;F)=&-\frac{e\kappa}{32\pi^2}\int_0^\infty\frac{dT}{T^3}\,{\rm e}^{-m^2T}\\
&\times\Big\{{\rm det}^{-\frac{1}{2}}\Big[\frac{\tan(\mathcal{Z})}{\mathcal{Z}}\Big]\int_0^T d\tau_1\int_0^T d\tau_2\,{\rm e}^{-k\cdot\overline{\mathcal{G}}_{B12}\cdot k}J^{\mu\nu,\alpha}_{\rm spin}+\frac{4}{3}iT^2eC^{\mu\nu,\alpha}\Big\}.
\end{aligned} \tag{4.9}$$

Once more this can still be simplified using the decompositions $\mathcal{G}_{B,F}=\mathcal{S}_{B,F}+\mathcal{A}_{B,F}$ where $\mathcal{S}^{\mu\nu}_{B,F}$ contains the even powers of $F^{\mu\nu}$ in the power series representation of $\mathcal{G}^{\mu\nu}_{B,F}$, and $\mathcal{A}^{\mu\nu}_{B,F}$ the odd ones. After this replacement all terms in the integrand are either symmetric or antisymmetric under the exchange $\tau_1 \leftrightarrow \tau_2$, and the antisymmetric ones can be deleted since their $\tau_{1,2}$ — integrals vanish. Further, as usual, we rescale $\tau_i = Tu_i, i = 1,2$, and set $u_2 = 0$. This leads to

$$
\begin{aligned}
\bar{\Pi}^{\mu\nu,\alpha}_{\rm spin}(k;F) = -\frac{e\kappa}{32\pi^2}\int_0^\infty \frac{dT}{T}\,\mathrm{e}^{-m^2T}\Bigg\{&\mathrm{det}^{-\frac{1}{2}}\left[\frac{\tan(\mathcal{Z})}{\mathcal{Z}}\right]\int_0^1 du_1 \\
&\times\ \mathrm{e}^{-k\cdot\overline{\mathcal{S}}_{B12}\cdot k}\sum_{m=1}^{3}\tilde{J}^{(\mu\nu),\alpha}_{\rm spin,m} + \frac{4}{3}ieC^{\mu\nu,\alpha}\Bigg\},
\end{aligned}
\tag{4.10}
$$

where

$$
\begin{aligned}
\tilde{J}^{\mu\nu,\alpha}_{\rm spin,1} =& \Big(\ddot{\mathcal{S}}^{\mu\nu}_{B11} - \dot{\mathcal{S}}^{\mu\nu}_{F11}\Big)\Big(k\cdot\,(\overline{\dot{\mathcal{A}}}_{B12} + \mathcal{A}_{F22})\Big)^{\alpha}, \\
\tilde{J}^{\mu\nu,\alpha}_{\rm spin,2} =& -\dot{\mathcal{S}}^{\mu\alpha}_{B12}\Big(\ddot{\mathcal{A}}_{B12}\cdot k\Big)^{\nu} + \mathcal{S}^{\mu\alpha}_{F12}\Big(\dot{\mathcal{A}}_{F12}\cdot k\Big)^{\nu} + \ddot{\mathcal{A}}^{\nu\alpha}_{B12}\Big(\,\dot{\mathcal{S}}_{B12}\cdot k\Big)^{\mu} - \dot{\mathcal{A}}^{\nu\alpha}_{F12}\Big(\mathcal{S}_{F12}\cdot k\Big)^{\mu} \\
&-\overline{\dot{\mathcal{A}}}^{\mu\alpha}_{B12}\Big(\ddot{\mathcal{S}}_{B12}\cdot k\Big)^{\nu} + \mathcal{A}^{\mu\alpha}_{F12}\Big(\dot{\mathcal{S}}_{F12}\cdot k\Big)^{\nu} \\
&+\ddot{\mathcal{S}}^{\nu\alpha}_{B12}\Big(\,(\overline{\dot{\mathcal{A}}}_{B12} + \mathcal{A}_{F11})\cdot k\Big)^{\mu} - \dot{\mathcal{S}}^{\nu\alpha}_{F12}\Big(\mathcal{A}_{F12}\cdot k\Big)^{\mu}, \\
\tilde{J}^{\mu\nu,\alpha}_{\rm spin,3} =& -\Big(\,\dot{\mathcal{S}}_{B12}\cdot k\Big)^{\mu}\Big[\Big(\,\dot{\mathcal{S}}_{B12}\cdot k\Big)^{\nu}\Big(k\cdot\,(\overline{\dot{\mathcal{A}}}_{B12} + \mathcal{A}_{F11})\Big)^{\alpha} - \Big(\,\mathcal{S}_{F12}\cdot k\Big)^{\nu}\Big(k\cdot\mathcal{A}_{F12}\Big)^{\alpha} \\
&+\Big(\,(\overline{\dot{\mathcal{A}}}_{B12} + \mathcal{A}_{F11})\cdot k\Big)^{\nu}\Big(k\cdot\,\dot{\mathcal{S}}_{B12}\Big)^{\alpha} - \Big(\mathcal{A}_{F12}\cdot k\Big)^{\nu}\Big(k\cdot\,\mathcal{S}_{F12}\Big)^{\alpha} \\
&\qquad\qquad -\overline{\dot{\mathcal{A}}}^{\nu\alpha}_{B12}k\cdot\dot{\mathcal{S}}_{B12}\cdot k + \mathcal{A}^{\nu\alpha}_{F12}k\cdot\mathcal{S}_{F12}\cdot k\Big] \\
&-\Big(\,\overline{\dot{\mathcal{A}}}_{B12}\cdot k\Big)^{\mu}\Big[\Big(\,\dot{\mathcal{S}}_{B12}\cdot k\Big)^{\nu}\Big(k\cdot\,\dot{\mathcal{S}}_{B12}\Big)^{\alpha} - \Big(\mathcal{S}_{F12}\cdot k\Big)^{\nu}\Big(k\cdot\,\mathcal{S}_{F12}\Big)^{\alpha} \\
&+\Big(\,(\overline{\dot{\mathcal{A}}}_{B12} + \mathcal{A}_{F11})\cdot k\Big)^{\nu}\Big(k\cdot\,(\overline{\dot{\mathcal{A}}}_{B12} + \mathcal{A}_{F11})\Big)^{\alpha} - \Big(\mathcal{A}_{F12}\cdot k\Big)^{\nu}\Big(k\cdot\,\mathcal{A}_{F12}\Big)^{\alpha} \\
&\qquad\qquad -\dot{\mathcal{S}}^{\nu\alpha}_{B12}k\cdot\dot{\mathcal{S}}_{B12}\cdot k + \mathcal{S}^{\nu\alpha}_{F12}k\cdot\mathcal{S}_{F12}\cdot k\Big].
\end{aligned}
\tag{4.11}
$$

It is then straightforward to write the integrands more explicitly using the decomposition formulas of appendix B.

The corresponding result for the scalar loop is obtained by omitting all terms involving the fermionic worldline Green's function, changing $\mathrm{det}^{-\frac{1}{2}}\left[\frac{\tan(\mathcal{Z})}{\mathcal{Z}}\right]$ to $\mathrm{det}^{-\frac{1}{2}}\left[\frac{\sin(\mathcal{Z})}{\mathcal{Z}}\right]$, and multiplying by a global factor of $-\frac{1}{2}$.

5 Tadpole contribution to electromagnetic photon-graviton conversion

The tadpole contribution (middle diagram in figure 2) can be constructed using our result (3.11) for the one-photon amplitude in the field, replacing k by k', ε by the υ of (2.13), and integrating over k'. This gives

$$
\begin{aligned}
\Gamma^{(\rm tadpole)}_{\rm spin}(k_0,\epsilon_0;k,\varepsilon;F) = 2ie\kappa\int_0^\infty dT(4\pi T)^{-\frac{D}{2}}\mathrm{e}^{-m^2T}\mathrm{det}^{\frac{1}{2}}\left[\frac{\mathcal{Z}}{\tan\mathcal{Z}}\right]\int d^Dk'\,\frac{\delta^D(k')}{k'^2} \\
\times k'\cdot\Big(\cot\mathcal{Z} + \tan\mathcal{Z} - \frac{1}{\mathcal{Z}}\Big)\cdot\{\epsilon_0, f\}\cdot k'\exp\left[\frac{T}{4}k'\cdot\Big(\frac{\cot\mathcal{Z}}{\mathcal{Z}} - \frac{1}{\mathcal{Z}^2}\Big)\cdot k'\right]
\end{aligned}
\tag{5.1}
$$

(we have also used the antisymmetry of the Lorentz matrix $\cot\mathcal{Z}+\tan\mathcal{Z}-\frac{1}{\mathcal{Z}}$). In the presence of the delta function, the factor $\exp[k'\cdots k']$ can be replaced by unity since the prefactor already contains two powers of k'. Applying the integral formula (1.3) leads to

$$\Gamma^{(\text{tadpole})}_{\text{spin}}(k_0=-k,\epsilon_0;k,\varepsilon;F)=\frac{4}{D}ie\kappa\int_0^\infty dT(4\pi T)^{-\frac{D}{2}}\,\mathrm{e}^{-m^2T}\\ \times\det{}^{\frac{1}{2}}\left[\frac{\mathcal{Z}}{\tan\mathcal{Z}}\right]\mathrm{tr}\left[\left(\cot\mathcal{Z}+\tan\mathcal{Z}-\frac{1}{\mathcal{Z}}\right)\cdot\epsilon_0\cdot f\right]. \tag{5.2}$$

This contribution, too, for $D=4$, contains a UV divergence, stemming from the terms linear in the field. Thus to calculate it we can replace the determinant factor with unity, and under the trace make the replacement

$$\cot\mathcal{Z}+\tan\mathcal{Z}-\frac{1}{\mathcal{Z}}\approx\frac{2}{3}\mathcal{Z}\,. \tag{5.3}$$

Extracting the pole in dimensional regularization we get

$$\Gamma^{(\text{tadpole})}_{\text{spin,div}}(k_0=-k,\epsilon_0;k,\varepsilon;F)=-\frac{4}{3}\frac{ie^2\kappa}{(4\pi)^2}\frac{1}{D-4}\mathrm{tr}\,(F\epsilon_0 f)\,. \tag{5.4}$$

As usual, we remove it by a subtraction at the integrand level, and the renormalized tadpole contribution becomes

$$\bar\Gamma^{(\text{tadpole})}_{\text{spin}}(k_0=-k,\epsilon_0;k,\varepsilon;F)=\frac{ie\kappa}{16\pi^2}\int_0^\infty\frac{dT}{T^2}\mathrm{e}^{-m^2T}\\ \times\left\{\det{}^{\frac{1}{2}}\left[\frac{\mathcal{Z}}{\tan\mathcal{Z}}\right]\mathrm{tr}\left[\left(\cot\mathcal{Z}+\tan\mathcal{Z}-\frac{1}{\mathcal{Z}}\right)\cdot\epsilon_0\cdot f\right]-\frac{2}{3}\mathrm{tr}\,(\mathcal{Z}\epsilon_0 f)\right\}. \tag{5.5}$$

Specializing to a constant magnetic background pointing along the z-axis (see appendix B), we rewrite the renormalized expression in D dimensions in order to perform the proper-time integral within dimensional regularization. Now the renormalized tadpole contribution is given by

$$\bar\Gamma^{(\text{tadpole})}_{\text{spin}}(k_0=-k,\epsilon_0;k,\varepsilon;B)=\frac{ie\kappa}{D}(2\pi)^{-\frac{D}{2}}\,\mathrm{tr}(r_+\epsilon_0 f)\int_0^\infty dT T^{-\frac{D}{2}}\,\mathrm{e}^{-m^2T}\\ \times\left\{\frac{z}{3}+\coth z-z\coth^2 z\right\} \tag{5.6}$$

with $z=eBT$ and $\mathcal{Z}=zr_+$. The z-integral can be carried out in closed form, leading to an expression of the renormalized tadpole contribution in terms of the Hurwitz zeta function $\zeta(s,a)$,

$$\bar\Gamma^{(\text{tadpole})}_{\text{spin}}(k_0=-k,\epsilon_0;k,\varepsilon;B)=-\frac{ie\kappa}{DeB}\left(\frac{\pi}{eB}\right)^{-\frac{D}{2}}\Gamma\left(1-\frac{D}{2}\right)\mathrm{tr}(r_+\epsilon_0 f)\\ \times\left[-2^{-\frac{D}{2}}\left(\frac{m^2}{eB}\right)^{\frac{D}{2}-1}-\frac{D}{2}\zeta\left(1-\frac{D}{2},1+\frac{m^2}{2eB}\right)\right.\\ \left.+\left(\frac{D}{2}-1\right)\frac{m^2}{2eB}\zeta\left(2-\frac{D}{2},1+\frac{m^2}{2eB}\right)+\frac{2^{1-\frac{D}{2}}}{3}\left(1-\frac{D}{2}\right)\left(\frac{m^2}{eB}\right)^{\frac{D}{2}-2}\right]. \tag{5.7}$$

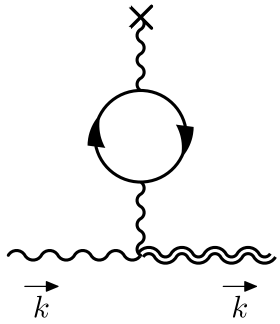

Figure 8. Tadpole diagram expanded to linear order in the field.

However, note that, unlike the renormalization which we performed on the irreducible contribution $\Gamma^{(\mathrm{irr})}_{\mathrm{spin}}$, this one is nothing new; the contribution of the tadpole linear in the field corresponds to the diagram shown in figure 8, which makes it clear that this subtraction is just a special case (the zero-momentum limit) of the QED photon wave function renormalization,

Note that, differently from the irreducible contribution, for the tadpole contribution the renormalization completely removes the part linear in the field, leaving only terms of cubic or higher order in the field.

Let us also give the corresponding result for the scalar loop:

$$\begin{aligned}\bar{\Gamma}^{(\mathrm{tadpole})}_{\mathrm{scal}}(k_0=-k,\epsilon_0;k,\varepsilon;F) &= -\frac{ie\kappa}{32\pi^2}\int_0^\infty \frac{dT}{T^2}\,\mathrm{e}^{-m^2T}\\ &\quad\times\left\{\mathrm{det}^{\frac{1}{2}}\left[\frac{\mathcal{Z}}{\sin\mathcal{Z}}\right]\mathrm{tr}\left[\left(\cot\mathcal{Z}-\frac{1}{\mathcal{Z}}\right)\cdot\epsilon_0\cdot f\right]+\frac{1}{3}\mathrm{tr}\left(\mathcal{Z}\epsilon_0 f\right)\right\}.\end{aligned} \tag{5.8}$$

Specializing to a purely magnetic field and $D=4$, eq. (5.8) becomes

$$\begin{aligned}\bar{\Gamma}^{(\mathrm{tadpole})}_{\mathrm{scal}} &= \frac{ie\kappa}{2D}(2\pi)^{-\frac{D}{2}}\,\mathrm{tr}\left(r_+\epsilon_0 f\right)\int_0^\infty dT\,T^{-\frac{D}{2}}\,e^{-m^2T}\\ &\quad\times\left\{\frac{z}{\sinh z}\left(\coth z-\frac{1}{z}\right)-\frac{z}{3}\right\}.\end{aligned} \tag{5.9}$$

Analogously to the spinor loop, one obtains the following representation for the scalar case:

$$\begin{aligned}\bar{\Gamma}^{(\mathrm{tadpole})}_{\mathrm{scal}}(k_0=-k,\epsilon_0;k,\varepsilon;F) &= -\frac{2ie\kappa}{DeB}\left(\frac{2\pi}{eB}\right)^{-\frac{D}{2}}\Gamma\left(1-\frac{D}{2}\right)\mathrm{tr}(r_+\epsilon_0 f)\\ &\quad\times\left[\frac{D}{2}\zeta\left(1-\frac{D}{2},\frac{1}{2}+\frac{m^2}{2eB}\right)+\left(1-\frac{D}{2}\right)\frac{m^2}{2eB}\zeta\left(2-\frac{D}{2},\frac{1}{2}+\frac{m^2}{2eB}\right)\right.\\ &\quad\left.+\frac{2^{-\frac{D}{2}}}{3}\left(1-\frac{D}{2}\right)\left(\frac{m^2}{eB}\right)^{\frac{D}{2}-2}\right].\end{aligned} \tag{5.10}$$

6 Third contribution to electromagnetic photon-graviton conversion

For the other reducible contribution, corresponding to the third diagram in figure 2, we can use eqs. (3.7), (3.8), with $k'=k_0=-k$, and our results for the vacuum polarization tensor in a constant field from subsection 3.3. Like the tadpole contribution, it has a UV divergence from the field-independent part of the vacuum polarization tensor. This part corresponds to the right-hand diagram of figure 2, but with the loop fermion taken in vacuum, and according to the LSZ formalism is not to be taken into account in the calculation of the on-shell amplitude. Using our result (3.23) for the on-shell renormalized one-loop vacuum polarization tensor in the constant field, and putting things together, we have

$$\bar{\Gamma}^{(\text{ext})}_{\text{spin}}(k_0 = -k, \epsilon_0; k, \varepsilon; F) = v_{F\beta}\Big(\bar{\Pi}^{\beta\alpha}_{\text{spin}}(k; F) - \bar{\Pi}^{\beta\alpha}_{\text{spin}}(k; 0)\Big)\varepsilon_\alpha \,, \tag{6.1}$$

with

$$v_F = i\kappa\frac{\{\epsilon_0, F\}\cdot k_0}{k_0^2} = -i\kappa\frac{\{\epsilon_0, F\}\cdot k}{k^2}\,, \tag{6.2}$$

$$\begin{aligned}\bar{\Pi}^{\beta\alpha}_{\text{spin}}(k; F) = \frac{e^2}{8\pi^2}\int_0^\infty \frac{dT}{T}\mathrm{e}^{-m^2T}\int_0^1 du_1\bigg\{&\det{}^{\frac{1}{2}}\left[\frac{\mathcal{Z}}{\tan\mathcal{Z}}\right]\tilde{I}^{\beta\alpha}_{\text{spin}}\;\mathrm{e}^{-Tk\cdot\Phi_{12}\cdot k}\\ &+\frac{2}{3}(\delta^{\beta\alpha}k^2 - k^\beta k^\alpha)\bigg\}\,,\end{aligned} \tag{6.3}$$

and $\tilde{I}^{\beta\alpha}_{\text{spin}}$ as given in (3.20). Note that this contribution, too, after renormalization starts with terms cubic in F, and that on-shell it has an IR divergence through the pole in (6.2). A proper treatment of the on-shell limit requires a somewhat different interpretation of the third diagram, namely as a correction to the tree-level amplitude due to the modified dispersion relation of the photon in the field [43]. In the presence of an external field, the physical photon eigenmodes no longer satisfy the vacuum on-shell condition $k^2 = 0$; instead, their propagation is governed by the dispersion relation $k^2 + \Pi(k) = 0$, where Π denotes the photon polarization tensor, leading in general to a field induced deviation from $k^2 = 0$.

The corresponding amplitude for a scalar loop is simply obtained by replacing $\bar{\Pi}^{\beta\alpha}_{\text{spin}}(k; F)$ with $\bar{\Pi}^{\beta\alpha}_{\text{scal}}(k; F)$ in (6.1).

7 Weak-field expansion of the tadpole contribution

Since phenomenological applications are primarily concerned with the regime of weak external fields, we briefly discuss the tadpole contribution in the limit of a weak magnetic background,

$$eB \ll m^2\,. \tag{7.1}$$

In this regime, the proper-time integrals admit a systematic asymptotic expansion in powers of the dimensionless ratio eB/m^2. To derive this expansion for the fermion QED case, it is convenient to return to the proper-time representation of the spinor tadpole contribution given in eq. (5.6), prior to performing the z-integration. Specializing to $D = 4$, the hyperbolic functions appearing in the integrand can be expanded for small

$$z = eBT\,, \tag{7.2}$$

which is justified since the proper-time integral is exponentially dominated by values $T \sim 1/m^2$ in the weak-field regime.

A compact representation of the weak-field expansion follows from the series representation of the integrand in eq. (5.6),

$$\frac{z}{3} + \coth z - z \coth^2 z = \sum_{n=2}^{\infty} \frac{2^{2n}\mathcal{B}_{2n}}{(2n-1)!}\, z^{2n-1}, \tag{7.3}$$

where $\mathcal{B}_n$ denotes the Bernoulli numbers. Substituting this expansion into the proper-time integral and carrying out the integration term by term yields the following closed-form weak-field expansion for the renormalized spinor tadpole contribution:

$$\begin{aligned}\bar{\Gamma}^{(\text{tadpole})}_{\text{spin}}(k_0 = -k, \epsilon_0; k, \varepsilon; B) = {}& \frac{ie^2\kappa B}{4}(2\pi)^{-2}\,\mathrm{tr}(r_+\epsilon_0 f) \\ & \times \sum_{n=2}^{\infty} \frac{2^{2n}\mathcal{B}_{2n}}{(2n-1)!}\left(\frac{eB}{m^2}\right)^{2n-2}\Gamma(2n-2)\,. \end{aligned} \tag{7.4}$$

The scalar loop can be treated in complete analogy. The small-z expansion of the integrand in eq. (5.9) reads

$$\frac{z}{\sinh z}\left(\coth z - \frac{1}{z}\right) - \frac{z}{3} = \sum_{n=2}^{\infty} \frac{(2^{2n}-2)}{(2n-1)!}\mathcal{B}_{2n}\, z^{2n-1}\,. \tag{7.5}$$

Proceeding exactly as in the spinor case, one finds the weak-field expansion for the renormalized scalar tadpole contribution in closed form as

$$\begin{aligned}\bar{\Gamma}^{(\text{tadpole})}_{\text{scal}}(k_0 = -k, \epsilon_0; k, \varepsilon; B) = {}& \frac{ie^2\kappa B}{4}(2\pi)^{-2}\,\mathrm{tr}(r_+\epsilon_0 f) \\ & \times \sum_{n=2}^{\infty} \frac{\left(2^{2n-1}-1\right)\mathcal{B}_{2n}}{(2n-1)!}\left(\frac{eB}{m^2}\right)^{2n-2}\Gamma(2n-2)\,. \end{aligned} \tag{7.6}$$

Comparing the coefficients of the weak-field expansions for the spinor and scalar loops, we see that the ratio of the coefficients multiplying the term of order $(eB/m^2)^{2n-2}$ is

$$\frac{\Gamma^{(n)}_{\text{spin}}}{\Gamma^{(n)}_{\text{scal}}} = \frac{2^{2n}}{2^{2n-1}-1}\,, \tag{7.7}$$

which in the large-n limit approaches 2.

8 Ward identities for the photon-graviton amplitude in a constant field

Finally, let us discuss how the Ward identities work for the photon-graviton amplitude in a constant field, returning to the off-shell unrenormalized amplitudes. Here there is no difference between the scalar and spinor-loop cases. It is easy to see that the gauge Ward identity holds for each of the three contributions separately, i.e.

$$\Gamma^{(\text{irr}),(\text{tadpole}),(\text{ext})}(-k, \epsilon_0; k, \delta\varepsilon; F) = 0\,, \tag{8.1}$$

where $\delta\varepsilon_i^\mu = k^\mu$.

For the gravitational Ward identity, defined by the transformation

$$\delta\epsilon_0^{\mu\nu} = k_0^\mu \zeta_0^\nu + \zeta_0^\mu k_0^\nu\,, \tag{8.2}$$

JHEP06(2026)175

a slight extension of the manipulations of the appendix of [20] shows that the inclusion of the one-photon amplitude leads to

$$\Gamma^{(\text{irr})}(k_0,\delta\epsilon_0;k,\varepsilon;F)=\Gamma(k_0+k,\tilde{\varepsilon};F)+\Gamma(k_0,\tilde{\varepsilon}_F;k,\varepsilon;F)\,, \tag{8.3}$$

where on the right-hand side we have now the one- and two-photon amplitudes, and

$$\begin{aligned}\tilde{\varepsilon}_i^{\mu} &\equiv \kappa(k_i^{\mu}\varepsilon_i^{\nu}-\varepsilon_i^{\mu}k_i^{\nu})\zeta_{0\nu}=\kappa(f_i\cdot\zeta_0)^{\mu}\,,\\ \tilde{\epsilon}_F^{\mu} &\equiv -i\kappa(F\cdot\zeta_0)^{\mu}\,.\end{aligned} \tag{8.4}$$

However, the first term on the right-hand side of (8.3) vanishes when setting $k_0=-k$. Using

$$\delta\upsilon_F^{\mu}=-\tilde{\varepsilon}_F^{\mu}-i\kappa\frac{k_0\cdot F\cdot\zeta_0}{k_0^2}k_0^{\mu}\,, \tag{8.5}$$

we can get from (8.3) the following on-shell Ward identity connecting $\Gamma^{(\text{irr})}_{\text{spin}}$ and $\Gamma^{(\text{ext})}_{\text{spin}}$,

$$\Gamma^{(\text{irr})}(-k,\delta\epsilon_0;k,\varepsilon;F)+\Gamma^{(\text{ext})}(-k,\delta\epsilon_0;k,\varepsilon;F)=0\,, \tag{8.6}$$

while the tadpole contribution on-shell becomes invariant by itself,

$$\Gamma^{(\text{tadpole})}(-k,\delta\epsilon_0;k,\varepsilon;F)=0\,, \tag{8.7}$$

as can be easily seen from (5.2) using the on-shell relations $k^2=\varepsilon\cdot k=\zeta_0\cdot k_0=0$.

9 Conclusions

In this work we have re-examined electromagnetic photon-graviton conversion in a constant external field within Einstein-Maxwell theory at the one-loop level. Although this conversion might eventually find applications in a broader range of physical contexts, here we have focused on its dichroism aspect, motivated by the analysis of ref. [43], which demonstrated that this process provides the leading Standard Model contribution to magnetic dichroism, despite its small absolute magnitude under realistic conditions.

Our main result is the identification and explicit evaluation of a non-vanishing tadpole contribution to the one-loop photon-graviton conversion amplitude, corresponding to the middle diagram in figure 2. Contrary to long-standing assumptions, this diagram does not vanish once the infrared behavior of the connecting photon propagator is treated correctly in dimensional regularization. Using the worldline formalism, we have provided a unified calculation of all one-loop contributions — irreducible, tadpole, and other reducible — for scalar and spinor loops in a general constant electromagnetic background.

Although the tadpole diagram contributes to the full one-loop amplitude, it is irrelevant for magnetic dichroism already at a structural level. From (5.6) we see that the magnetic tadpole contribution depends on the photon and graviton polarizations through a global factor $\text{tr}(F\epsilon_0 f)$, which (for on-shell gravitons) is proportional to the tree-level amplitude $\Gamma^{(\text{tree})}(k_0,\epsilon_0;k,\varepsilon;F)$ of eq. (1.1), implying identical photon-graviton conversion rates for the two photon polarizations. Consequently, the magnetic dichroism remains unchanged, and the analysis of Ahlers et al. [43] is not affected by the presence of the tadpole diagram.

In the weak-field regime $eB/m^2 \ll 1$, relevant for phenomenological applications, the tadpole contribution is moreover parametrically suppressed compared to the irreducible one-loop contribution. In the strong-field limit $eB/m^2 \gg 1$, the relative importance of different one-loop contributions can change qualitatively, as has been demonstrated in related contexts [70]. In such regimes, tadpole-type contributions may play a more prominent role in the overall amplitude, which could be relevant not only from a theoretical standpoint but also in extensions of the Standard Model involving light or weakly charged particles, such as mini-charged particles [71], for which strong-field effects can become physically relevant.

Independently of these phenomenological considerations, the inclusion of the tadpole diagram is conceptually important. It completes the one-loop description of photon-graviton conversion in external fields and clarifies its close connection to other one-particle reducible effects in background-field QED. Our results thus provide a consistent and complete framework for future studies, including strong-field backgrounds, non-standard charged sectors, and higher-loop corrections.

A Einstein-Maxwell theory

In our euclidean conventions, the action for a scalar field coupled to electromagnetism and gravity is

$$S[\phi,\phi^*;g,A] = \int d^Dx\sqrt{g}\Big[g^{\mu\nu}(\partial_\mu - ieA_\mu)\phi^*(\partial_\nu + ieA_\nu)\phi + (m^2+\xi R)\phi^*\phi\Big]\,. \tag{A.1}$$

In the spin 1/2 case, the euclidean action for a Dirac field Ψ coupled to electromagnetism and gravity is given by

$$S[\Psi,\bar\Psi;e,A] = \int d^Dx\,\sqrt{g}\,\bar\Psi(\nabla\!\!\!\!/ + m)\Psi \tag{A.2}$$

where the vielbein e^a_μ and the spin connection $\omega_{\mu ab}$ appear inside the Dirac operator

$$\nabla\!\!\!\!/ = \gamma^a e_a^\mu \nabla_\mu\,, \qquad \nabla_\mu = \partial_\mu + ieA_\mu + \frac{1}{4}\omega_{\mu ab}\gamma^a\gamma^b\,. \tag{A.3}$$

Moreover, we will need also the interaction between photons and gravitons, which is described by the action of Einstein-Maxwell theory (in euclidean conventions)

$$S[g,A] = \int d^Dx\,\sqrt{g}\left(-\frac{1}{2\kappa^2}R + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) \tag{A.4}$$

where $\kappa^2 = 8\pi G_N$. Let us expand this action to quadratic order in the fields. Expanding $g_{\mu\nu} = \delta_{\mu\nu} + \kappa h_{\mu\nu}$ and $A_\mu = \bar A_\mu + a_\mu$, so that $F_{\mu\nu} = \bar F_{\mu\nu} + f_{\mu\nu}$, and using the short-hand notation $h \equiv \delta^{\mu\nu}h_{\mu\nu}$, one obtains the following quadratic approximation in the fluctuations

$(h_{\mu\nu}, a_\mu)$ around the background $(\delta_{\mu\nu}, \bar{A}_\mu)$:

$$\begin{aligned} S_{(2)} = \int d^D x \Big\{ &\frac{1}{4}(\partial_\alpha h_{\mu\nu})^2 - \frac{1}{8}(\partial_\alpha h)^2 - \frac{1}{2}\Big(\partial^\alpha h_{\alpha\mu} - \frac{1}{2}\partial_\mu h\Big)^2 + \frac{1}{2}(\partial_\mu a_\nu)^2 - \frac{1}{2}(\partial^\mu a_\mu)^2 \\ &- \frac{\kappa}{2} h_{\mu\nu}\left(\bar{F}^{\mu\alpha}\bar{F}^\nu{}_\alpha - \frac{1}{4}\delta^{\mu\nu}\bar{F}^2\right) \\ &- \kappa h_{\mu\nu}\left(\bar{F}^{\mu\alpha} f^\nu{}_\alpha - \frac{1}{4}\delta^{\mu\nu}\bar{F}^{\alpha\beta} f_{\alpha\beta}\right) \\ &+ \frac{\kappa^2}{4}\left[\left(\frac{1}{8}h^2 - \frac{1}{4}h^2_{\mu\nu}\right)\bar{F}^2 + h^{\mu\nu}h^{\alpha\beta}\bar{F}_{\mu\alpha}\bar{F}_{\nu\beta} + (2h^{\mu\alpha}h^\nu{}_\alpha - hh^{\mu\nu})\bar{F}_{\mu\beta}\bar{F}_\nu{}^\beta\right]\Big\} . \end{aligned} \tag{A.5}$$

In the second line of this expression, we recognize the linear coupling $-\frac{\kappa}{2}h_{\mu\nu}\bar{T}^{\mu\nu}$ of the graviton $h_{\mu\nu}$ to the stress tensor of the background electromagnetic field

$$\bar{T}^{\mu\nu} = \bar{F}^{\mu\alpha}\bar{F}^\nu{}_\alpha - \frac{1}{4}\delta^{\mu\nu}\bar{F}^2 . \tag{A.6}$$

The existence of this tadpole vertex indicates that the nontrivial background stress tensor tends to curve space. The third line in (A.5) gives instead the tree-level graviton-photon conversion in the electromagnetic background. Using plane waves

$$h_{\mu\nu}(x) = \epsilon_{0\mu\nu}\, \mathrm{e}^{ik_0 x} , \qquad a_\alpha(x) = \varepsilon_\alpha\, \mathrm{e}^{ikx} \tag{A.7}$$

we get for this conversion term the vertex (deleting now the 'bar' on F)

$$\Delta S_{(2)} = (2\pi)^D \delta^D(k_0 + k)\, \epsilon_{0\mu\nu}\varepsilon_\alpha\, (-i\kappa)\left[F^{\mu\alpha}k^\nu - (F\cdot k)^\mu\delta^{\nu\alpha} + \frac{1}{2}\delta^{\mu\nu}(F\cdot k)^\alpha\right] .$$

The two-point functions (which we denote by Π), in either coordinate or momentum space, are contained in $S_{(2)}$ (or in the quadratic part of the full effective action $\Gamma_{(2)} = -S_{(2)} + \Gamma^{(1-\text{loop})}_{(2)} + \cdots$) as follows[1]

$$\begin{aligned} S_{(2)} &= \int d^D x \Big\{ \frac{1}{2}h_{\mu\nu}(x)\Pi^{\mu\nu,\lambda\rho}(-i\partial)h_{\lambda\rho}(x) + \frac{1}{2}a_\alpha(x)\Pi^{\alpha\beta}(-i\partial)a_\beta(x) \\ &\qquad + h_{\mu\nu}(x)\Pi^{\mu\nu,\alpha}(-i\partial)a_\alpha(x) - \frac{\kappa}{2}h_{\mu\nu}(x)\bar{T}^{\mu\nu}(x)\Big\} \\ &= \int \frac{d^D k}{(2\pi)^D}\Big\{\frac{1}{2}h_{\mu\nu}(-k)\Pi^{\mu\nu,\lambda\rho}(k)h_{\lambda\rho}(k) + \frac{1}{2}a_\alpha(-k)\Pi^{\alpha\beta}(k)a_\beta(k) \\ &\qquad + h_{\mu\nu}(-k)\Pi^{\mu\nu,\alpha}(k)a_\alpha(k) - \frac{\kappa}{2}h_{\mu\nu}(-k)\bar{T}^{\mu\nu}(k)\Big\} . \end{aligned} \tag{A.8}$$

The equations of motion in terms of these two-point functions then read

$$\delta a_\alpha(-k): \qquad \Pi^{\alpha\beta}(k)a_\beta(k) + \Pi^{\mu\nu,\alpha}(-k)h_{\mu\nu}(k) = 0 \tag{A.9}$$

$$\delta h_{\mu\nu}(-k): \qquad \Pi^{\mu\nu,\lambda\rho}(k)h_{\lambda\rho}(k) + \Pi^{\mu\nu,\alpha}(k)a_\alpha(k) = \frac{\kappa}{2}\bar{T}^{\mu\nu}(k) \tag{A.10}$$

[1] We Fourier transform fields as $h_{\mu\nu}(k) = \int d^D x\, \mathrm{e}^{-ikx}\, h_{\mu\nu}(x)$, with inverse Fourier transform given by $h_{\mu\nu}(x) = \int \frac{d^D k}{(2\pi)^D} h_{\mu\nu}(k)\, \mathrm{e}^{ikx}$.

JHEP06(2026)175

and, in particular, one obtains from (A.5)

$$\Pi^{\alpha\beta}_{\rm tree}(k) = k^2\delta^{\alpha\beta} - k^\alpha k^\beta\,, \tag{A.11}$$

$$\Pi^{\mu\nu,\alpha}_{\rm tree}(k) = -\frac{i\kappa}{2}C^{\mu\nu,\alpha} \tag{A.12}$$

with

$$C^{\mu\nu,\alpha} = F^{\mu\alpha}k^\nu + F^{\nu\alpha}k^\mu - (F\cdot k)^\mu\delta^{\nu\alpha} - (F\cdot k)^\nu\delta^{\mu\alpha} + (F\cdot k)^\alpha\delta^{\mu\nu} \tag{A.13}$$

depicted in figure 1.

From the cubic fluctuations, we will still need the term coupling the graviton to two photons:

$$S^{G\gamma\gamma}_{(3)} = -\frac{\kappa}{2}\int d^Dx\, h_{\mu\nu}\left(f^{\mu\alpha}f^\nu{}_\alpha - \frac{1}{4}\delta^{\mu\nu}f^2\right). \tag{A.14}$$

B Worldline Green's functions and determinants in a constant field

Define $\mathcal{Z}_{\mu\nu} \equiv eF_{\mu\nu}T$.

B.1 Periodic case

Determinant factor:

$$\det{}^{\frac{1}{2}}\left[\frac{\mathcal{Z}}{\sin\mathcal{Z}}\right]. \tag{B.1}$$

Worldline Green's function and its derivatives:

$$\mathcal{G}_{B12} = \frac{T}{2\mathcal{Z}^2}\left(\frac{\mathcal{Z}}{\sin\mathcal{Z}}\,{\rm e}^{-i\mathcal{Z}\dot G_{B12}} + i\mathcal{Z}\dot G_{B12} - 1\right), \tag{B.2}$$

$$\dot{\mathcal{G}}_{B12} = \frac{i}{\mathcal{Z}}\left(\frac{\mathcal{Z}}{\sin\mathcal{Z}}\,{\rm e}^{-i\mathcal{Z}\dot G_{B12}} - 1\right), \tag{B.3}$$

$$\ddot{\mathcal{G}}_{B12} = 2\delta_{12} - \frac{2}{T}\frac{\mathcal{Z}}{\sin\mathcal{Z}}\,{\rm e}^{-i\mathcal{Z}\dot G_{B12}} \tag{B.4}$$

Symmetry properties:

$$\mathcal{G}_{B12} = \mathcal{G}^T_{B21}, \quad \dot{\mathcal{G}}_{B12} = -\dot{\mathcal{G}}^T_{B21}, \quad \ddot{\mathcal{G}}_{B12} = \ddot{\mathcal{G}}^T_{B21}. \tag{B.5}$$

Coincidence limits:

$$\mathcal{G}_B(\tau,\tau) = \frac{T}{2}\left(\frac{\cot\mathcal{Z}}{\mathcal{Z}} - \frac{1}{\mathcal{Z}^2}\right), \tag{B.6}$$

$$\dot{\mathcal{G}}_B(\tau,\tau) = i\cot\mathcal{Z} - \frac{i}{\mathcal{Z}}, \tag{B.7}$$

$$\ddot{\mathcal{G}}_B(\tau,\tau) = 2\delta(0) - \frac{2}{T}\mathcal{Z}\cot\mathcal{Z}\,. \tag{B.8}$$

Weak-field expansion:

$$\mathcal{G}_{B12} = G_{B12} - \frac{1}{6} - \frac{i}{3}\dot G_{B12}G_{B12}\mathcal{Z} + \left(\frac{1}{3T}G^2_{B12} - \frac{T}{90}\right)\mathcal{Z}^2 + O(\mathcal{Z}^3)\,, \tag{B.9}$$

$$\dot{\mathcal{G}}_{B12} = \dot G_{B12} + 2i\left(\frac{G_{B12}}{T} - \frac{1}{6}\right)\mathcal{Z} + \frac{2}{3T}\dot G_{B12}G_{B12}\mathcal{Z}^2 + O(\mathcal{Z}^3)\,, \tag{B.10}$$

$$\ddot{\mathcal{G}}_{B12} = 2\delta_{12} - \frac{2}{T} + \frac{2}{T}i\dot G_{B12}\mathcal{Z} - 4\left(\frac{G_{B12}}{T} - \frac{1}{6}\right)\mathcal{Z}^2 + O(\mathcal{Z}^3)\,. \tag{B.11}$$

Even-odd decomposition: For the purpose of simplifying integrals, it will often be convenient to decompose the constant-field worldline Green's functions as

$$\mathcal{G}_{B12} = \mathcal{S}_{B12} + \mathcal{A}_{B12} \tag{B.12}$$

where $\mathcal{S}_B$ denotes the even and $\mathcal{A}_B$ the odd part as a power series in F. Note that $\mathcal{S}_B$ is symmetric as a Lorentz matrix and $\mathcal{A}_B$ antisymmetric, which motivates our notation.

Special constant fields:

- **Parallel electric and magnetic fields:**

  in the generic case, it is possible to find a Lorentz system where $\mathbf{E}$ and $\mathbf{B}$ both point along the z — axis. If we further assume, for definiteness, that $\mathbf{E}\cdot\mathbf{B} > 0$, then we can take them both to point along the positive z-axis, $\mathbf{E} = (0,0,E), \mathbf{B} = (0,0,B)$. The euclidean field strength tensor then takes the form

$$F = \begin{pmatrix} 0 & B & 0 & 0 \\ -B & 0 & 0 & 0 \\ 0 & 0 & 0 & iE \\ 0 & 0 & -iE & 0 \end{pmatrix} . \tag{B.13}$$

  Defining $z_+ \equiv eBT$ and $z_- \equiv ieET$, the determinant factor (B.1) can then be written as

$$\det{}^{-\frac{1}{2}}\left[\frac{\sin\mathcal{Z}}{\mathcal{Z}}\right] = \frac{z_+ z_-}{\sinh z_+ \sinh z_-} = \frac{eBT}{\sinh eBT}\frac{eET}{\sin eET} . \tag{B.14}$$

  We further define Lorentz matrices g_+, g_-, and r_+, r_- by

$$g_+ \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad g_- \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$r_+ \equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad r_- \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{B.15}$$

  Then one can show the following matrix decompositions of the worldline Green's functions

$$\mathcal{S}^{\mu\nu}_{B12} = -\frac{T}{2}\sum_{\alpha=\pm}\frac{A_{B12}(z_\alpha)}{z_\alpha}\,g^{\mu\nu}_\alpha , \tag{B.16}$$

$$\mathcal{A}^{\mu\nu}_{B12} = \frac{iT}{2}\sum_{\alpha=\pm}\frac{S_{B12}(z_\alpha) - \dot{G}_{B12}}{z_\alpha}\,r^{\mu\nu}_\alpha , \tag{B.17}$$

$$\dot{\mathcal{S}}^{\mu\nu}_{B12} = \sum_{\alpha=\pm} S_{B12}(z_\alpha)\,g^{\mu\nu}_\alpha , \tag{B.18}$$

$$\dot{\mathcal{A}}^{\mu\nu}_{B12} = -i \sum_{\alpha=\pm} A_{B12}(z_\alpha)\, r^{\mu\nu}_\alpha \,, \tag{B.19}$$

$$\ddot{\mathcal{S}}^{\mu\nu}_{B12} = \ddot{G}_{B12}\delta^{\mu\nu} - \frac{2}{T} \sum_{\alpha=\pm} z_\alpha A_{B12}(z_\alpha)\, g^{\mu\nu}_\alpha \,, \tag{B.20}$$

$$\ddot{\mathcal{A}}^{\mu\nu}_{B12} = \frac{2i}{T} \sum_{\alpha=\pm} z_\alpha S_{B12}(z_\alpha)\, r^{\mu\nu}_\alpha \,. \tag{B.21}$$

These decompositions involve essentially only the two scalar, dimensionless functions S_{B12} and A_{B12},

$$S_{B12}(z) = \frac{\sinh(z\,\dot{G}_{B12})}{\sinh z} \,, \tag{B.22}$$

$$A_{B12}(z) = \frac{\cosh(z\,\dot{G}_{B12})}{\sinh z} - \frac{1}{z} \,. \tag{B.23}$$

Of those only A_{B12} has a non-vanishing coincidence limit,

$$A_{Bii} = \coth z - \frac{1}{z} \,. \tag{B.24}$$

Let us also write down the first few terms of the weak field expansions of these functions,

$$S_{B12}(z) = \dot{G}_{B12}\Big[1 - \frac{2}{3}\frac{G_{B12}}{T}z^2 + \Big(\frac{2}{45}\frac{G_{B12}}{T} + \frac{2}{15}\frac{G^2_{B12}}{T^2}\Big)z^4 + \mathrm{O}(z^6)\Big] \,, \tag{B.25}$$

$$A_{B12}(z) = \Big(\frac{1}{3} - 2\frac{G_{B12}}{T}\Big)z + \Big(-\frac{1}{45} + \frac{2}{3}\frac{G^2_{B12}}{T^2}\Big)z^3 + \mathrm{O}(z^5) \,. \tag{B.26}$$

- **The purely magnetic field case:**

  We let the magnetic field point along the positive z-axis and denote $z = eBT$. Then the determinant factor (B.1) becomes

$$\det{}^{-\frac{1}{2}}\Big[\frac{\sin\mathcal{Z}}{\mathcal{Z}}\Big] = \frac{z}{\sinh z} \,. \tag{B.27}$$

  The decompositions (B.16)–(B.21) now involve only the three matrices g_+, g_- and r_+:

$$\mathcal{G}_B(\tau_1,\tau_2) = G_{B12}\, g_- - \frac{T}{2z}A_{B12}(z)g_+ + \frac{T}{2z}\Big(S_{B12}(z) - \dot{G}_{B12}\Big)ir_+ \,, \tag{B.28}$$

$$\dot{\mathcal{G}}_B(\tau_1,\tau_2) = \dot{G}_{B12}\, g_- + S_{B12}(z)g_+ - A_{B12}(z)ir_+ \,, \tag{B.29}$$

$$\ddot{\mathcal{G}}_B(\tau_1,\tau_2) = \ddot{G}_{B12}\mathbb{1} - \frac{2z}{T}A_{B12}(z)g_+ + \frac{2z}{T}S_{B12}(z)ir_+ \,. \tag{B.30}$$

  The coincidence limits (B.6)–(B.8) turn into

$$\mathcal{G}_B(\tau,\tau) = -\frac{T}{6}\, g_- - \frac{T}{2z}\Big(\coth z - \frac{1}{z}\Big)g_+ = -\frac{T}{6}\mathbb{1} - \frac{T}{2z}\Big(\coth z - \frac{1}{z} - \frac{z}{3}\Big)g_+ \,, \tag{B.31}$$

$$\dot{\mathcal{G}}_B(\tau,\tau) = -\Big(\coth z - \frac{1}{z}\Big)ir_+ \,, \tag{B.32}$$

$$\ddot{\mathcal{G}}_B(\tau,\tau) = 2\Big(\delta(0) - \frac{1}{T}\Big)\mathbb{1} - \frac{2z}{T}\Big(\coth z - \frac{1}{z}\Big)g_+ \,. \tag{B.33}$$

- **The purely electric field case:**

  To obtain the analogue of the magnetic formulas (B.27)–(B.33) for a purely electric field pointing into the z-direction, just replace $z = eBT$ by $z = ieET$ and interchange $+ \leftrightarrow -$ everywhere.

- **The crossed-field case:**

  In a "crossed field", defined by $\mathbf{E} \perp \mathbf{B}, E = B$, both invariants $B^2 - E^2$ and $\mathbf{E} \cdot \mathbf{B}$ vanish. For such a field $F^3 = 0$, so that the power series (B.9)–(B.11) break off after their quadratic terms. The worldline correlators thus can be represented by the terms given there.

  The determinant factor becomes unity.

B.2 Anti-periodic case

Determinant factor:

$$\det{}^{\frac{1}{2}}\left[\cos \mathcal{Z}\right] . \tag{B.34}$$

Worldline Green's function and its derivative:

$$\mathcal{G}_F(\tau_1, \tau_2) = G_{F12} \frac{\mathrm{e}^{-i\mathcal{Z}\dot{G}_{B12}}}{\cos\mathcal{Z}} , \tag{B.35}$$

$$\dot{\mathcal{G}}_F(\tau_1, \tau_2) = 2\delta_{12} + \frac{2i}{T} G_{F12} \frac{\mathcal{Z}}{\cos \mathcal{Z}} \mathrm{e}^{-i\mathcal{Z}\dot{G}_{B12}} . \tag{B.36}$$

Symmetry properties:

$$\mathcal{G}_{F12} = -\mathcal{G}_{F21}^T , \quad \dot{\mathcal{G}}_{F12} = \dot{\mathcal{G}}_{F21}^T . \tag{B.37}$$

Coincidence limits:

$$\mathcal{G}_F(\tau, \tau) = -i \tan\mathcal{Z} , \tag{B.38}$$

$$\dot{\mathcal{G}}_F(\tau, \tau) = 2\delta(0) + \frac{2}{T} \mathcal{Z} \tan \mathcal{Z} . \tag{B.39}$$

Weak-field expansion:

$$\mathcal{G}_{F12} = G_{F12} - iG_{F12}\dot{G}_{B12}\mathcal{Z} + 2G_{F12}\frac{G_{B12}}{T}\mathcal{Z}^2 + O(\mathcal{Z}^3) , \tag{B.40}$$

$$\dot{\mathcal{G}}_{F12} = 2\delta_{12} + \frac{2}{T} i\mathcal{Z} G_{F12} + \frac{2}{T} G_{F12}\dot{G}_{B12}\mathcal{Z}^2 + O(\mathcal{Z}^3) . \tag{B.41}$$

Even-odd decomposition:

$$\mathcal{G}_F = \mathcal{S}_F + \mathcal{A}_F . \tag{B.42}$$

Special constant fields:

- **Parallel electric and magnetic fields:**

  With the same notation as in the periodic case above,

$$\det{}^{\frac{1}{2}}[\cos\mathcal{Z}] = \cosh z_+ \cosh z_- = \cosh(eBT)\cos(eET)\,, \tag{B.43}$$

$$\mathcal{S}^{\mu\nu}_{F12} = \sum_{\alpha=\pm} S_{F12}(z_\alpha)\, g^{\mu\nu}_\alpha\,, \tag{B.44}$$

$$\mathcal{A}^{\mu\nu}_{F12} = -\sum_{\alpha=\pm} A_{F12}(z_\alpha)\, ir^{\mu\nu}_\alpha\,, \tag{B.45}$$

$$\dot{\mathcal{S}}^{\mu\nu}_{F12} = \dot{G}_{F12}\mathbb{1} - \frac{2}{T}\sum_{\alpha=\pm} A_{F12}(z_\alpha)\, g^{\mu\nu}_\alpha\,, \tag{B.46}$$

$$\dot{\mathcal{A}}^{\mu\nu}_{F12} = \frac{2}{T}\sum_{\alpha=\pm} z_\alpha S_{F12}(z_\alpha)\, ir^{\mu\nu}_\alpha\,. \tag{B.47}$$

  The scalar, dimensionless coefficient functions appearing in these formulas are

$$S_{F12}(z) = G_{F12}\frac{\cosh(z\,\dot{G}_{B12})}{\cosh(z)}\,, \tag{B.48}$$

$$A_{F12}(z) = G_{F12}\frac{\sinh(z\,\dot{G}_{B12})}{\cosh(z)}\,. \tag{B.49}$$

  Of those, it is A_{F12} that has a non-vanishing coincidence limit

$$A_{Fii} = \tanh z\,. \tag{B.50}$$

  As for the periodic case above, let us also write down the first few terms of the weak field expansions of these functions:

$$S_{F12}(z) = G_{F12}\Bigg[1 - 2\frac{G_{B12}}{T}z^2 + \frac{2}{3}\left(\frac{G_{B12}}{T} + \frac{G^2_{B12}}{T^2}\right)z^4 + \mathrm{O}(z^6)\Bigg]\,, \tag{B.51}$$

$$A_{F12}(z) = G_{F12}\dot{G}_{B12}\Bigg[z - \left(\frac{1}{3} + \frac{2}{3}\frac{G_{B12}}{T}\right)z^3 + \mathrm{O}(z^5)\Bigg]\,. \tag{B.52}$$

- **The purely magnetic field case:**

  With the same notations as for the periodic case above, the determinant factor becomes

$$\det{}^{\frac{1}{2}}[\cos\mathcal{Z}] = \cosh z\,. \tag{B.53}$$

  The worldline correlators (B.35), (B.36) specialize to

$$\mathcal{G}_F(\tau_1,\tau_2) = G_{F12}\,g_- + S_{F12}(z)g_+ - A_{F12}(z)\,ir_+\,, \tag{B.54}$$

$$\dot{\mathcal{G}}_F(\tau_1,\tau_2) = \dot{G}_{F12}\mathbb{1} - \frac{2z}{T}A_{F12}(z)g_+ + \frac{2z}{T}S_{F12}(z)\,ir_+ \tag{B.55}$$

  with coincidence limits

$$\mathcal{G}_F(\tau,\tau) = -\tanh z\,ir_+\,, \tag{B.56}$$

$$\dot{\mathcal{G}}_F(\tau,\tau) = 2\delta(0)\mathbb{1} - \frac{2z}{T}\tanh z\,g_+\,. \tag{B.57}$$

- **The purely electric field case:**

  The rules for the transition from the magnetic to the electric field are the same as for the periodic case above.

- **The crossed-field case:**

  As in the periodic case, for a crossed field the determinant factor becomes unity, and the worldline correlators are given exactly by their weak-field expansion to order $O(F^2)$, (B.40) and (B.41).

Acknowledgments

We would like to thank F. Fillion-Gourdeau and A. Ilderton for helpful discussions and correspondence.

Data Availability Statement. This article has no associated data or the data will not be deposited.

Code Availability Statement. This article has no associated code or the code will not be deposited.

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